

Macroeconomics II
Dynamic macroeconomics
Class 1: Introduction and first models

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1 Class 1: introduction and first models

What we will do today

1. Organization of course
 - (a) Parallel course on Matlab (Francisco)
 - (b) Structure of classes
 - (c) Exercises each week (to be turned in)
 - (d) Final exam (take home)
2. Today's class
 - (a) Definition of our topic
 - (b) Correction of data (Hordrick-Prescott filter)
 - (c) First model (Solow)
 - i. build model
 - ii. study stationary states
 - iii. make log-linear version to study stochastic properties

Dynamic Macroeconomics

Why dynamic models

- Much of what we are interested in in economics occurs over time
 - inflation
 - growth
 - asset pricing

- dynamic effects of government policies
- models with stochastic elements
- Various models for handling this
 - Older "Keynesian" reduced form models
 - Models that can be resolved as a single recursive problem
 - * via Bellman equations or variational methods
 - Models that need to be approximated (why?)
 - * linear approximations
 - * quadratic or higher approximations
- Need modeling techniques and solution techniques

Dynamic Macroeconomics

RBC models

1. Based on growth models (Solow)
2. Kydland and Prescott (won Nobel prize for this)
 - (a) Their model is quite complicated
 - (b) Ours will be based on model by Hansen
3. We use lots of other models
 - Overlapping generations
 - Models of money
 - Credit markets
 - Restrictions in prices and wages
 - Open economy models.

1.1 What are RBC models

What are RBC models?

1. Micro-foundations based models
2. Full specification of
 - (a) preferences
 - (b) production technologies
 - (c) budget constraints
3. Dynamic

- (a) Infinite horizon
- (b) Rational expectations
- 4. Solve linear or quadratic approximation of model
 - (a) Dynamic programming (Bellmans equation)
 - (b) Log-linearization of model
 - (c) Linear quadratic techniques
- 5. Get time path of economy

1.2 What is a successful RBC model

A successful RBC model

1. Captures the second moments of the economy
2. Captures correlation with output
3. Mimics the impulse-response functions of the economy
4. Rules of Prescott
 - (a) One needs to know the question one wants to ask
 - (b) Model should be designed for that question
 - (c) Model should use off-the-shelf technology
 - (d) Model should be explicitly dynamic

.What data do we use

1. Model is compared to the data
 - (a) How well do second moments match those of data
 - (b) Compare impulse response of model to those of data
2. Data is normally filtered
 - (a) Models are solved around stationary state
 - (b) Mimic stationary state using filter
 - i. Remove trends
 - ii. Remove long run growth
3. Hodrick-Prescott Filter is frequently used

1.3 Hodrick-Prescott filter

Hodrick-Prescott Filter I

1. Want to find **economic cycles**
2. Times series data can be seen as of the form

$$Y_{t+1} = \alpha Y_t + \beta t + \varepsilon_t$$

3. α is the auto-regressive coefficient
4. β is the linear trend (one could add quadratic trends, etc.)
5. What is the value of α ?
 - (a) If $|\alpha| \geq 1$ time path is explosive (called unit root when $|\alpha| = 1$)
 - (b) if $|\alpha| < 1$ time path stable
 - i. $\alpha^n \rightarrow 0$ as $n \rightarrow \infty$

Hodrick-Prescott Filter II

1. One wants to remove trends
2. Formula for H-P filter

$$\min_{\{u_t\}} \frac{1}{T} \sum_{t=1}^T (y_t - u_t)^2 + \frac{\lambda}{T} \sum_{t=2}^{T-1} \left[(u_{t+1} - u_t) - (u_t - u_{t-1}) \right]^2$$

3. The sequence $\{u_t\}$ is the 'trend'
4. $\frac{1}{T} \sum_{t=1}^T (y_t - u_t)^2$ is sum of squared errors
5. $\frac{\lambda}{T} \sum_{t=2}^{T-1} \left[(u_{t+1} - u_t) - (u_t - u_{t-1}) \right]^2$ is a discrete version of the second derivative
 - (a) λ is parameter that controls smoothing
 - (b) H-P recommend $\lambda = 1600$
6. The detrended data (used for models) is $y_t - u_t$

Example of HP filter on Argentina GDP data

Filtered data with lambda = 1600

Problems with H-P filter

1. Not sure what it means
 - (a) Can change characteristics of time series process

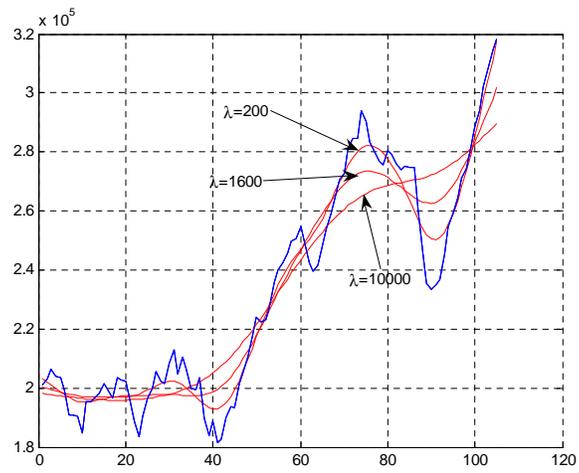


Figure 1: Three Hodrick-Prescott filters

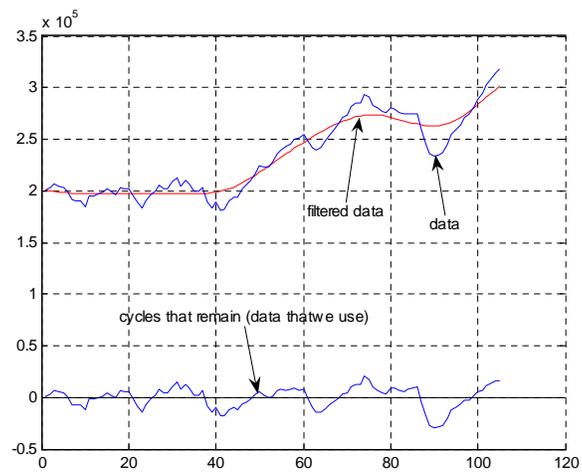


Figure 2: GDP data, HP trend, and detrended GDP data

- (b) Removes certain (low) frequencies from the process
 - (c) What does the data explain after the filtering?
2. Endpoint problems
 - (a) Because it is a weighed average of points
 - (b) At endpoints, only have one side of average
 3. However, has become common practice to use H-P filter

1.4 Solow's growth model

Solow's growth model

1. First model we will study
2. Basis for RBC and New Keynesian models
3. Touch some of the basics of course
 - (a) Micro foundations
 - (b) Generates time path
 - (c) Can find linear version
 - (d) See how much of the cycle the **model** explains

Reference: Robert Solow: A Contribution to the Theory of Economic Growth (1956)

Solow's growth model

Production function

$$Y_t = A_t F(K_t, N_t)$$

Y_t = output, K_t = stock of capital, N_t = labor, $A_t = (1 + \alpha)^t A_0$ = the level of technology

Properties of $F(\cdot)$

twice continuously differentiable

homogenous of degree 1

$$\beta F(K_t, N_t) = F(\beta K_t, \beta N_t)$$

increasing in both arguments

Strictly concave: $F_i > 0$, $F_{ii} < 0$, $F_{ij} > 0$

Inada conditions: $F(K_t, 0) = 0$, $F(0, N_t) = 0$, $F_N(K_t, 0) = \infty$, $F_K(0, N_t) = \infty$, $F_N(K_t, \infty) = 0$, $F_K(\infty, N_t) = 0$

Solow model: population growth

Constant net growth rate of population (and labor) = λ

The labor supply follows the rule:

$$N_{t+1} = (1 + \lambda) N_t$$

Solow model: Capital accumulation process

Capital follows the accumulation rule

$$K_{t+1} = (1 - \delta) K_t + I_t$$

I_t is the investment at time t

Savings rule

Savings is a fixed fraction of output (major simplification of Solow)

$$S_t = sY_t$$

- No household optimization for savings decision
- the parameter s is given exogenously

Equilibrium conditions for investment-savings

$$S_t = I_t$$

The full model

$$Y_t = A_t F(K_t, N_t)$$

$$K_{t+1} = (1 - \delta) K_t + I_t$$

$$S_t = sY_t$$

$$S_t = I_t$$

$$N_{t+1} = (1 + \lambda) N_t$$

The first four can be combined to give the equation

$$K_{t+1} = (1 - \delta) K_t + sA_t F(K_t, N_t)$$

Put everything into 'per unit of labor' terms

Divide both sides of

$$K_{t+1} = (1 - \delta) K_t + sA_t F(K_t, N_t)$$

by $N_{t+1} = (1 + \lambda) N_t$ to get

$$\frac{K_{t+1}}{N_{t+1}} = \frac{(1 - \delta) K_t}{(1 + \lambda) N_t} + \frac{sA_t F(K_t, N_t)}{(1 + \lambda) N_t}$$

Defining per-worker terms: $k_t = K_t/N_t$, $y_t = Y_t/N_t$, this can be written as

$$k_{t+1} = \frac{(1 - \delta)}{(1 + \lambda)}k_t + \frac{sA_t}{(1 + \lambda)} \frac{F(K_t, N_t)}{N_t}$$

Because the production function is homogenous of degree 1,

$$\frac{F(K_t, N_t)}{N_t} = F\left(\frac{K_t}{N_t}, \frac{N_t}{N_t}\right) = F(k_t, 1) \equiv f(k_t),$$

The Solow difference equation can be written as

$$k_{t+1} = G(k_t) = \frac{(1 - \delta)}{(1 + \lambda)}k_t + \frac{sA_t}{(1 + \lambda)}f(k_t)$$

We will (in class) let $A_t = A_0$, a constant technology

- Model with constant technological growth

$$A_t = (1 + \alpha)^t A_0$$

where $\alpha = 0$

- Using the Cobb-Douglas production function and zero technological growth
 $f(k_t) = A_0 k_t^\theta$
- The Solow first difference equation is

$$k_{t+1} = \widehat{G}(k_t) = \frac{(1 - \delta)}{(1 + \lambda)}k_t + \frac{sA_0}{(1 + \lambda)}k_t^\theta$$

- Given some initial k_0 , the time path for the economy can be found
- Where the solution k to the equation

$$k = \widehat{G}(k) = \frac{(1 - \delta)}{(1 + \lambda)}k + \frac{sA_0}{(1 + \lambda)}k^\theta$$

is a stationary state capital stock for the economy

1.5 Stability conditions for Solow Model

Solow Graph

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Stability conditions for Solow model

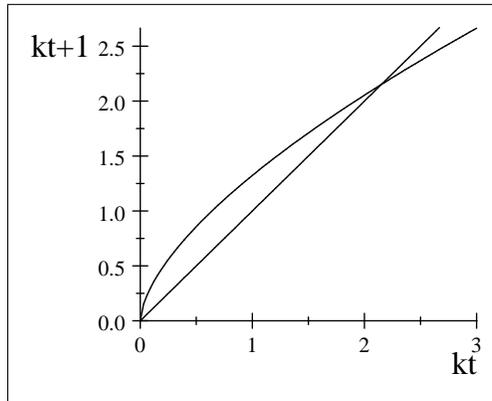


Figure 3: Stability conditions

There are two stationary states

zero

at

$$k = \left(\frac{sA_0}{\delta + \lambda} \right)^{\frac{1}{1-\theta}}$$

this last is an attractor

A result of Solow Model

Poor countries (those with less capital) grow faster than rich

Gross growth rate of capital can be written as

$$\gamma_t = \frac{k_{t+1}}{k_t} = \frac{(1 - \delta)k_t + sA_0f(k_t)}{(1 + \lambda)k_t}.$$

And taking the derivative with respect to capital:

$$\frac{d\gamma_t}{dk_t} = \frac{sA_0}{(1 + \lambda)k_t^2} [f'(k_t)k_t - f(k_t)] < 0$$

when $k_t > 0$.

Countries with higher k_t have smaller growth rates of capital

Stochastic variables

- A probability space: (Ω, F, P)
 - Ω is a set that contains all possible states of nature that might occur
 - * What is a state of nature
 - F is a collection of subsets of Ω
 - * Why do we need this?
 - P is a probability measure over F
- Some draw from Ω is observed each period

2 A stochastic Solow Model

A stochastic Solow Model

Suppose that technology is stochastic,

$$A_t = \bar{A}e^{\varepsilon_t}$$

- \bar{A} is positive
- random term ε_t has a normal distribution with mean zero
- So in stationary state (with $\varepsilon_t = 0$), $e^{\varepsilon_t} = 1$
- e^{ε_t} is never negative

The stochastic version of the Solow first order difference equation is

$$k_{t+1} = \frac{(1 - \delta)k_t + s\bar{A}e^{\varepsilon_t}f(k_t)}{(1 + \lambda)}$$

Version in Logs (model is simpler this way)

1. Divide both sides by k_t
2. replace k_{t+1}/k_t by growth rate γ_t
3. There is a problem with $\ln((1 - \delta)k_t + s\bar{A}e^{\varepsilon_t}f(k_t))$
4. Take logs of both sides

$$\ln \left[\gamma_t - \frac{1 - \delta}{1 + \lambda} \right] = \ln \frac{s\bar{A}}{(1 + \lambda)} + \ln \frac{f(k_t)}{k_t} + \varepsilon_t.$$

5. Use Cobb-Douglas production function, $f(k_t) = k_t^\theta$, to get

$$\ln \left[\gamma_t - \frac{1 - \delta}{1 + \lambda} \right] = \varphi - (1 - \theta) \ln k_t + \varepsilon_t,$$

where $\varphi \equiv \ln [s\bar{A}/(1 + \lambda)]$

Simulation of stochastic Solow model

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Parameters used

$\bar{A} = 1$

$s = .2$

$\lambda = .02$

$\delta = .1$

$\theta = .36$

$\sigma_\varepsilon^2 = .2$

Log-linear version of (no growth trend) Solow model

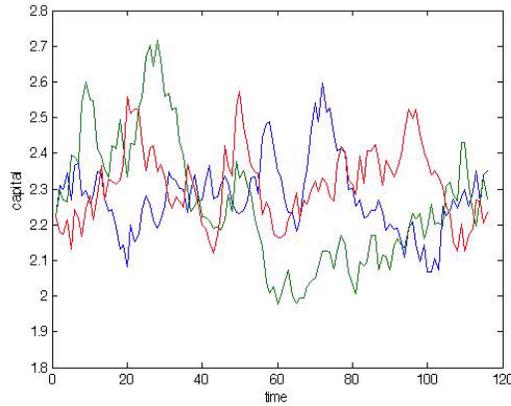


Figure 4: Three simulations of the exact Solow model

1. We have a second way to handle dynamics
2. Make an (log) linear approximation of the model
3. Study the dynamics of this linear model
4. Exists technology for solving dynamic linear models

Log-linear version of (no growth trend) Solow model

1. Begin with the Solow difference equation

$$(1 + \lambda) k_{t+1} = (1 - \delta)k_t + s\bar{A}e^{\varepsilon_t} k_t^\theta,$$

2. Define $\tilde{k}_t = \ln(k_t - \bar{k})$, where \bar{k} is the stationary state value of capital
3. Then $k_t = \bar{k}e^{\tilde{k}_t}$
4. The Solow difference equation is now

$$(1 + \lambda) \bar{k}e^{\tilde{k}_{t+1}} = (1 - \delta)\bar{k}e^{\tilde{k}_t} + s\bar{A}e^{\varepsilon_t}\bar{k}^\theta e^{\theta\tilde{k}_t},$$

5. Use the approximation that for small \tilde{k}_t ,

$$e^{a\tilde{k}_t} = 1 + a\tilde{k}_t$$

6. This approximation comes from a first order Taylor expansion of $e^{a\tilde{k}_t}$

Log-linear Solow model

1. The approximation of

$$(1 + \lambda) \bar{k} \tilde{k}_{t+1} = (1 - \delta) \bar{k} \tilde{k}_t + s \bar{A} \bar{k}^\theta e^{\theta \tilde{k}_t + \varepsilon^t},$$

(note that $e^{\varepsilon^t} e^{\theta \tilde{k}_t} = e^{\theta \tilde{k}_t + \varepsilon^t}$) is

$$(1 + \lambda) \bar{k} \left(1 + \tilde{k}_{t+1}\right) = (1 - \delta) \bar{k} \left(1 + \tilde{k}_t\right) + s \bar{A} \bar{k}^\theta \left(1 + \theta \tilde{k}_t + e^{\varepsilon^t}\right).$$

2. The 1's in parenthesis drop out because

$$(1 + \lambda) \bar{k} = (1 - \delta) \bar{k} + s \bar{A} \bar{k}^\theta.$$

3. The log-linear version of the model is

$$(1 + \lambda) \bar{k} \tilde{k}_{t+1} = \left[(1 - \delta) \bar{k} + s \bar{A} \bar{k}^\theta \theta\right] \tilde{k}_t + s \bar{A} \bar{k}^\theta e^{\varepsilon^t}.$$

Log-linear Solow model (continued)

1. This further simplifies (since $\delta + \lambda = s \bar{A} \bar{k}^{\theta-1}$) to

$$\tilde{k}_{t+1} = \frac{1 - \delta + (\delta + \lambda) \theta}{1 + \lambda} \tilde{k}_t + \frac{\delta + \lambda}{1 + \lambda} e^{\varepsilon^t}.$$

2. Notice that

$$\frac{(1 - \delta) + (\delta + \lambda) \theta}{1 + \lambda} = \frac{1 + \lambda \theta - \delta(1 - \theta)}{1 + \lambda} < 1$$

3. Let $B = \frac{1 + \lambda \theta - \delta(1 - \theta)}{1 + \lambda}$ and $C = \frac{\delta + \lambda}{1 + \lambda} < 1$ and time $t + 1$ capital is equal to

$$\tilde{k}_{t+1} = C \sum_{i=0}^{\infty} B^i \varepsilon_{t-i}$$

This is a convergent sequence

Variance of capital

The variance of time $t + 1$ capital is equal to

$$\begin{aligned} \text{var}(\tilde{k}_{t+1}) &= E(\tilde{k}_{t+1})^2 = E\left(C \sum_{i=0}^{\infty} B^i \varepsilon_{t-i}\right) \left(C \sum_{i=0}^{\infty} B^i \varepsilon_{t-i}\right) \\ &= C^2 E\left(\sum_{i=0}^{\infty} B^i \varepsilon_{t-i}\right) \left(\sum_{i=0}^{\infty} B^i \varepsilon_{t-i}\right) \end{aligned}$$

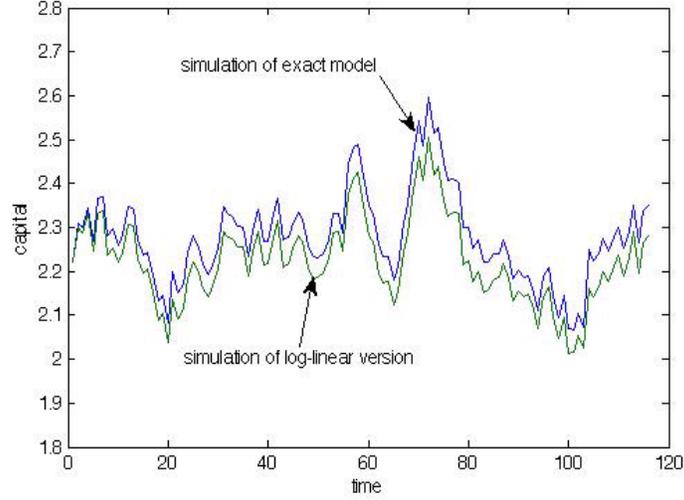


Figure 5: Simulations of the exact and log-linear Solow model

Since the shocks are independent, $E(\varepsilon_i \varepsilon_j) = 0$ if $i \neq j$, and $= \sigma^2$ if $i = j$. All cross terms drop out and the sums become

$$\text{var}(\tilde{k}_{t+1}) = C^2 \sigma^2 E \sum_{i=0}^{\infty} B^{2i} = \frac{C^2}{1 - B^2} \sigma^2$$

Using the parameters given before, $B = 0.97176$ and $C = .044118$ so that

$$\text{var}(\tilde{k}_{t+1}) = \frac{(.044118)^2}{1 - (0.97176)^2} \sigma^2 = .024955 \sigma^2$$

The variance of capital is very small compared to the technology shock variance

.Time path of capital (comparing approximate to exact models)

.Variance of output

1. Output comes from the equation

$$y_t = A_t k_t^\theta = \bar{A} e^{\varepsilon_t} k_t^\theta$$

2. A log-linear approximation of this is

$$\bar{y}(1 + \tilde{y}_t) \approx \bar{y} e^{\tilde{y}_t} = \bar{A} k^\theta e^{\theta \tilde{k}_t + \varepsilon_t} \approx \bar{A} k^\theta (1 + \theta \tilde{k}_t + \varepsilon_t)$$

which simplifies to

$$\tilde{y}_t \approx \theta \tilde{k}_t + \varepsilon_t$$

The variance of output is almost entirely made up of variance in technology.

The effect of changes in the capital stock are very small

Solow model does not explain the persistence of shocks (cycles) in an economy

Homework

1. Usando un modelo de Solow con crecimiento en tecnología de

$$A_t = (1 + \alpha)^t A_0$$

con $A_0 = 5$ y $\alpha = .01$, encuentre los caminos de capital y de producto per capita por 25 periodo cuando $k(0) = .1$ y cuando $k(0) = 40$. Grafice los caminos de capital Usa $s = .15$, $\theta = .57$, y $\delta = .018$. (Nota: debe usar Matlab para esta problema).

2. Muestre los graficos de los las tasas de crecimiento de capital en problema 1.

3. Describe the time path for a Solow model where there is an externality for each firm and each firm has the production function

$$y_t^j = A_t k_t^\theta h_t^{1-\theta} K_t^{1-\theta}.$$

Lower case letters are for the firm and upper case letters are for the economy as a whole. The total capital in the economy functions as an externality for each firm.