Macroeconomics II
OLG models: the savings decision

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1 Overlapping generations models

.Overlapping generations models
Adding individual savings decisions
Why study these models
The savings rate s is no longer a constant
It is determined by optimizing behavior of individuals
These models are sometimes included inside RBC models

References:
Peter Diamond (1965) "National debt in a neoclassical growth model,"AER p.1126-1150.
McCandless and Wallace (1991) Introduction to Dynamic Macroeconomic Theory, Chapter 9

1.1 Basics of the model

.The Environment

- Time is discrete: $t = 0, 1, 2, 3, 4, ...$
- Individuals live 2 periods
  - Generation t is born in period t
  - Lives in periods t and $t + 1$
  - They are dead in periods $t + 2$ and onwards
  - There are $N(t)$ members of generation t
- Individuals have preferences: person h of generation t

$$u_t^h(c_t^h(t), c_t^h(t + 1))$$

- Individuals have a labor endownment:

- no labor supply decision, they supply all

$$h_t^h = \left[h_t^h(t), h_t^h(t+1)\right]$$

.The Environment

- Production technology

$$Y(t) = F(K(t), H(t))$$

- Total labor supplied in period t

$$H(t) = \sum_{h=1}^{N(t)} h_t^h(t) + \sum_{h=1}^{N(t-1)} h_{t-1}^h(t)$$

Useful to defind labor supplied by each generation in period t
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$$H_t(t) = \sum_{h=1}^{N(t)} h_t^h(t)$$

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$$H_{t-1}(t) = \sum_{h=1}^{N(t-1)} h_{t-1}^h(t)$$

- Feasiblity constraint for period t

$$Y(t) = F(K(t), H(t)) \geq \sum_{h=1}^{N(t)} c_t^h(t) + \sum_{h=1}^{N(t-1)} c_{t-1}^h(t) + K(t+1)$$

.The economy (a market economy)

- Budget constraint of generation t when young

$$w_t h_t^h(t) = c_t^h(t) + l^h(t) + k^h(t+1)$$

  - w_t are the wages paid at time t
  - $l^h(t)$ is lending or borrowing (if negative) of young in period t
  - Can only lend to own generation (why?)
  - so

$$0 = \sum_{h=1}^{N(t)} l_h(t)$$

- Budget constaint of generation t when old

$$c_t^h(t+1) = w_{t+1}h_t^h(t+1) + r_t l^h(t) + rental_{t+1}k^h(t+1)$$

  - r_t is the interest rate on private loans
  - $rental_{t+1}$ is the rental rate on capital when it is being used
  - we assume that capital depreciates 100% in the period of use

Factor market conditions

- Perfectly compeditive factor markets
- So

$$w_t = F_H(K(t), H(t))$$

  and

$$rental_t = F_K(K(t), H(t))$$

- Factor rentals equal their marginal products

Lifetime budget constraint and Arbitrage conditions

- The lifetime budget constraint is

$$c_t^h(t) + \frac{c_t^h(t+1)}{r_t} = w_t h_t^h(t) + \frac{w_{t+1}h_t^h(t+1)}{r_t} - k^h(t+1)\left[1 - \frac{rental_{t+1}}{r_t}\right]$$

- Arbitrage condition (actually NO arbitrage condition)

$$rental_{t+1} = r_t$$

  - What if not true: two options $rental_{t+1} < r_t$ or $rental_{t+1} > r_t$
  - If $rental_{t+1} < r_t$
    * Everyone wants to lend and no one holds capital
    * But marginal product of capital $\longrightarrow \infty$ when $K(t) \longrightarrow 0$
    * Not an equilibrium
  - If $rental_{t+1} > r_t$
    * Everyone wants to borrow an infinite amount to buy capital
    * This can't be an equilibrium
  - Only $rental_{t+1} = r_t$ remains as possibility

Lifetime budget constraint II

- The lifetime budget constraint simplifies to

$$c_t^h(t) + \frac{c_t^h(t+1)}{r_t} = w_t h_t^h(t) + \frac{w_{t+1}h_t^h(t+1)}{r_t}$$

- The present value of lifetime consumption equals the present value of lifetime wage income

1.2 Definition of an equilibrium

Definition of an equilibrium

A *competitive equilibrium* is a sequence of prices, $\{w_t, rental_t, r_t\}_{t=0}^{\infty}$, and quantities, $\left\{\{c_t^h(t)\}_{h=1}^{N(t)}, \{c_{t-1}^h(t)\}_{h=1}^{N(t-1)}, K(t+1)\right\}_{t=0}^{\infty}$, such that each member h of each generation $t > 0$ maximizes the utility function,

$$u_t^h(c_t^h(t), c_t^h(t+1))$$

subject to the lifetime budget constraint,

$$c_t^h(t) + \frac{c_t^h(t+1)}{r_t} = w_t h_t^h(t) + \frac{w_{t+1} h_t^h(t+1)}{r_t}$$

and the equilibrium conditions,

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$$\begin{aligned} r_t &= rental_{t+1} \\ w_t &= F_H(K(t), H(t))) \\ rental_t &= F_K(K(t), H(t)) \end{aligned}$$

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$$H(t) = \sum_{h=1}^{N(t)} h_t^h(t) + \sum_{h=1}^{N(t-1)} h_{t-1}^h(t)$$

hold in every period.

How to solve: Individual problem

- Substitute the budget constraints into the utility function to get

$$\max_{c_t^h(t)} u(c_t^h(t), r_t w_t h_t^h(t) + w_{t+1} h_t^h(t+1) - r_t c_t^h(t))$$

- First order condtions are

$$\begin{aligned} & u_1(c_t^h(t), r_t w_t h_t^h(t) + w_{t+1} h_t^h(t+1) - r_t c_t^h(t)) \\ = \ & r_t u_2(c_t^h(t), r_t w_t h_t^h(t) + w_{t+1} h_t^h(t+1) - r_t c_t^h(t)), \end{aligned}$$

- One can solve for a savings functions as

$$s_t^h(w_t, w_{t+1}, r_t) = l^h(t) + k^h(t+1).$$

How to solve: Aggregating savings functions

- Sum savings across members of generation t:

$$S_t(\cdot) = \sum_{h=1}^{N(t)} s_t^h(\cdot) = \sum_{h=1}^{N(t)} l^h(t) + \sum_{h=1}^{N(t)} k^h(t+1).$$

- In equilibrium, total borrowing and lending among generation t is

$$\sum_{h=1}^{N(t)} l^h(t) = 0,$$

- Definition of aggregate capital is

$$K(t+1) = \sum_{h=1}^{N(t)} k^h(t+1),$$

- So the aggregate savings equation is

$$S_t(w_t, w_{t+1}, r_t) = K(t+1).$$

How to solve: Getting a first order difference equation

- From the factor markets we have wages and rentals in terms of capital and labor,

$$\begin{aligned} w_t &= F_H(K(t), H(t))) \\ rental_t &= F_K(K(t), H(t)) \end{aligned}$$

- Substitute these into the aggregate saving equation to get (this ugly equation)

$$S_t(F_H(K(t), H(t)), F_H(K(t+1), H(t+1)), F_K(K(t+1), H(t+1)))$$
$$= K(t+1)$$

- This can be simplified (since $H(t)$ and $H(t+1)$ are constants) to get a function of the form

$$K(t+1) = G(K(t))$$

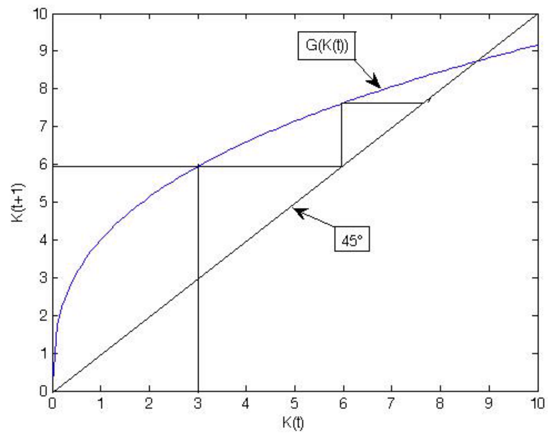
An example economy

- Let the utility function be

$$u_t^h = u(c_t^h(t), c_t^h(t+1)) = c_t^h(t) c_t^h(t+1)^\beta$$

and the production function be

$$Y_t = F(K(t), H(t)) = K(t)^\theta H(t)^{1-\theta}$$



- After a bunch of algebra,

$$K(t+1) = G(K(t)) = \frac{\theta\beta\frac{H_t(t)}{H(t)^\theta}}{\left[\frac{H_t(t+1)}{H(t+1)}\right] + \frac{\theta(1+\beta)}{(1-\theta)}} K(t)^\theta = \kappa K(t)^\theta$$

  where κ is a constant equal to

$$\kappa = \frac{\theta\beta\frac{H_t(t)}{H(t)^\theta}}{\left[\frac{H_t(t+1)}{H(t+1)}\right] + \frac{\theta(1+\beta)}{(1-\theta)}}$$

.The $K(t+1) = G(K(t))$ function is
.Extensions

- The book contains an example of how to make the model stochastic
- One might try making a log-linear version of the model and looking for the second moment properties
- Exercise 1 Work out the details of finding the savings function given above.
- Exercise 2 Find the equilibrium in a constant population economy where, in each period, the government imposes a small tax, t, on each young person and gives that amount to each old person. Both the young and the old see these taxes as lump sum.