Macroeconomics II
OLG suplemental material
Working out the example economy

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.Consumption function

- Given the utility function

$$c_t^h(t)c_t^h(t+1)^\beta$$

and the lifetime budget constraint

$$c_t^h(t+1) = r_t w_t h_t^h(t) + w_{t+1} h_t^h(t+1) - r_t c_t^h(t),$$

- the maximization problem can be written as

$$c_t^h(t)\left[r_t w_t h_t^h(t) + w_{t+1} h_t^h(t+1) - r_t c_t^h(t)\right]^\beta$$

taking the derivative with respect to $c_t^h(t)$ gives

$$\left[r_t w_t h_t^h(t) + w_{t+1} h_t^h(t+1) - r_t c_t^h(t)\right]^\beta = \beta r_t c_t^h(t)\left[r_t w_t h_t^h(t) + w_{t+1} h_t^h(t+1) - r_t c_t^h(t)\right]^{\beta-1}$$

.Consumption function

- This long equation simplifies to

$$r_t w_t h_t^h(t) + w_{t+1} h_t^h(t+1) - r_t c_t^h(t) = \beta r_t c_t^h(t)$$

and we get the consumption function

$$\begin{aligned} c_t^h(t) &= \frac{r_t w_t h_t^h(t) + w_{t+1} h_t^h(t+1)}{(\beta+1)\, r_t} \\ &= \frac{w_t h_t^h(t)}{(\beta+1)} + \frac{w_{t+1} h_t^h(t+1)}{(\beta+1)\, r_t} \end{aligned}$$

Savings function

- Savings is defined as

$$s_t^h = w_t h_t^h(t) - c_t^h(t)$$

- so

$$s_t^h = w_t h_t^h(t) - \frac{w_t h_t^h(t)}{(\beta+1)} - \frac{w_{t+1} h_t^h(t+1)}{(\beta+1)\, r_t} = \frac{\beta w_t h_t^h(t)}{(\beta+1)} - \frac{w_{t+1} h_t^h(t+1)}{(\beta+1)\, r_t}$$

Aggregate savings

- Summing over the $N(t)$ members of generation t, we get

$$\begin{aligned} S_t(\cdot) &\equiv \sum_{h=1}^{N(t)} s_t^h(\cdot) \\ &= \sum_{h=1}^{N(t)} \left[\frac{\beta w_t h_t^h(t)}{(\beta+1)} - \frac{w_{t+1} h_t^h(t+1)}{(\beta+1)\, r_t} \right] \\ S_t &= \frac{\beta w_t H_t(t)}{(\beta+1)} - \frac{w_{t+1} H_t(t+1)}{(\beta+1)\, r_t} \end{aligned}$$

Equilibrium

- In equilibrium,

$$S_t = \frac{\beta w_t H_t(t)}{(\beta+1)} - \frac{w_{t+1} H_t(t+1)}{(\beta+1)\, r_t} = K(t+1)$$

  and

$$r_t = rental_{t*1}$$

  where

$$rental_t = \theta K(t)^{\theta-1} H(t)^{1-\theta}$$

  and

$$w_t = (1-\theta)\, K(t)^{\theta} H(t)^{-\theta}$$

Equilibrium

- Substituting the wage and rental conditions into the $S_t = K(t+1)$ equilibrium condition gives

$$\begin{aligned} &\frac{\beta\,(1-\theta)\, K(t)^{\theta} H(t)^{-\theta} H_t(t)}{(\beta+1)} - \frac{(1-\theta)\, K(t+1)^{\theta} H(t+1)^{-\theta} H_t(t+1)}{(\beta+1)\,\theta K(t+1)^{\theta-1} H(t+1)^{1-\theta}} \\ = \;& K(t+1) \end{aligned}$$

  and this simplifies to

$$K(t+1) = \frac{\theta\beta \frac{H_t(t)}{H(t)^{\theta}}}{\left[\frac{H_t(t+1)}{H(t+1)}\right] + \frac{\theta(1+\beta)}{(1-\theta)}} K(t)^{\theta}$$