Macroeconomics II
Infinitely lived agents: Variational methods

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1 Infinitely lived agents

Infinitely lived agents

- Why do we use this kind of model
- Do we believe people plan over infinite horizons
  - Many people do have children
  - They care about the children
  - Maybe plan for their dynasty
- Normally, far into the future matters little
  - and we don't know when we will die
- These problems are **Mathematically tractable**
- We first do simple Robinson Crusoe model
  - Why is the Robinson Crusoe novel interesting to economists?
- Then economy with multiple agents and markets

Robinson Crusoe economy

- Only one agent (who we call Robinson Crusoe) with utility

$$\sum_{i=0}^{\infty} \beta^i u(c_{t+i})$$

- Notice that it is a discounted infinite horizon utility
- Budget constaints

$$k_{t+1} = (1-\delta)k_t + i_t$$

and

$$y_t = f(k_t) \geq c_t + i_t$$

1.1 Solving for stationary state

.Variational methods

Solving for a stationary state

1. Assume that values for endogenous variables for periods $s-1$ and $s+1$ are known
2. Solve for values in period s
3. Find equilibrium where values for $s-1$, s, and $s+1$ are the same
4. This is a stationary state that meets the equilibrium conditions

.General framework

- Optimization problem ($F\left(x_{t+i}, x_{t+1+i}\right)$ is objective function)

$$\max_{\{x_x\}_{s=t}^{\infty}} \sum_{i=0}^{\infty} \beta^i F\left(x_{t+i}, x_{t+1+i}\right).$$

- The Euler condition (a necessary condition)

$$0 = F_2\left(x_{s-1}, x_s\right) + \beta F_1\left(x_s, x_{s+1}\right)$$

- Transversality condition

$$\lim_{s \to \infty} \beta^s F_1\left(x_s, x_{s+1}\right) x_s = 0.$$

.Example economy

1. Substitute budget constraints into utility function. Get:

$$\sum_{i=0}^{\infty} \beta^i u(f(k_{t+i}) - k_{t+i+1} + (1-\delta)k_{t+i})$$

2. First order conditions are

$$\begin{aligned} 0 &= \beta^{s-t} u'(f(k_s) - k_{s+1} + (1-\delta)k_s)\left(f'(k_s) + (1-\delta)\right) \\ &\quad - \beta^{s-t-1} u'(f(k_{s-1}) - k_s + (1-\delta)k_{s-1}). \end{aligned}$$

.Example economy (continued)

1. Which can be written as the Euler equation (a necessary condition)

$$f'(k_s) + (1-\delta) = \frac{u'(f(k_{s-1}) - k_s + (1-\delta)k_{s-1})}{\beta u'(f(k_s) - k_{s+1} + (1-\delta)k_s)}.$$

2. In a stationary state, $k_{s+1} = k_s = k_{s-1} = \overline{k}$, so the above becomes

$$f'(\overline{k}) = \frac{1}{\beta} - 1 + \delta$$

The value of $\overline{k}$ is implicitly defined

3. Savings in a stationary state is $\overline{s} = \delta\overline{k}$

- See book for Robinson Crusoe with labor decision included

.Transversality condition

- Transversality condition for example economy

$$\lim_{i\to\infty} \beta^i u'\left(f(k_{t+i}) - k_{t+i+1} + (1-\delta)k_{t+i}\right)\left(f'(k_{t+i}) + (1-\delta)\right)k_{t+i} = 0.$$

- In a stationary state with $\overline{k} > 0$

$$\begin{aligned} & \lim_{i\to\infty} \beta^i u'\left(f(\overline{k}) - \delta\overline{k}\right)\left(f'(\overline{k}) + (1-\delta)\right)\overline{k} \\ = & \lim_{i\to\infty} \beta^i u'\left(f(\overline{k}) - \delta\overline{k}\right)\left(\frac{1}{\beta} - 1 + \delta + (1-\delta)\right)\overline{k} \\ = & \lim_{i\to\infty} \beta^{i-1} u'\left(f(\overline{k}) - \delta\overline{k}\right)\overline{k} \to 0 \end{aligned}$$

- because $u'\left(f(\overline{k}) - \delta\overline{k}\right)\overline{k}$ is a finite constant for $\overline{k} > 0$

1.2 Market economy

.Competitive market economy

- Many agents
  - Households
  - Firms
- Markets for goods, labor, and rental of capital
- Individual take aggregate values as given
- Decentralized decision making

.Competitive market economy
Unit mass of agents
indexed from 0 to 1 continuously

$$H_t = \int_0^1 h_t^i di \text{ and } K_t = \int_0^1 k_t^i di$$

Include individual labor supply decision

$$u\left(c_t^i, l_t^i\right) = u\left(c_t^i, 1 - h_t^i\right)$$

subject to the budget constaints

$$c_t^i = w_t h_t^i + r_t k_t^i + i_t^i$$
$$k_{t+1}^i = (1-\delta)\, k_t^i + i_t^i$$
$$w_t = f_h\left(K_t, H_t\right)$$
$$r_t = f_h\left(K_t, H_t\right)$$

Set up problem as Legrange problem

$$\begin{aligned} L &= \sum_{t=0}^{\infty} \beta^t \left[u\left(c_t^i, 1-h_t^i\right) - \lambda_t^1 \left(k_{t+1}^i - (1-\delta)\, k_t^i - i_t^i\right) \right. \\ & \left. -\lambda_t^2 \left(f_h\left(K_t, H_t\right) h_t^i + f_k\left(K_t, H_t\right) k_t^i - c_t^i - i_t^i\right)\right] \end{aligned}$$

- Difference between h_t^i and H_t and k_t^i and K_t
- In optimization, H_t and K_t are considered given
- Equilibrium condition are applied after optimization

First order conditions

$$\begin{aligned} \frac{\partial L}{\partial c_t^i} &= u_c\left(c_t^i, 1-h_t^i\right) + \lambda_t^2 = 0 \\ \frac{\partial L}{\partial h_t^i} &= -u_h\left(c_t^i, 1-h_t^i\right) - \lambda_t^2 f_h\left(K_t, H_t\right) = 0 \\ \frac{\partial L}{\partial k_{t+1}^i} &= -\lambda_t^1 + \beta \lambda_{t+1}^1 (1-\delta) - \beta \lambda_{t+1}^2 f_k\left(K_{t+1}, H_{t+1}\right) = 0 \\ \frac{\partial L}{\partial i_t^i} &= \lambda_t^1 = \lambda_t^2 = 0 \end{aligned}$$

First order conditions (continued)
These simplify to

$$\begin{aligned} u_c\left(c_t^i, 1-h_t^i\right) &= \lambda_t^1 = -\lambda_t^2 \\ \frac{u_h\left(c_t^i, 1-h_t^i\right)}{u_c\left(c_t^i, 1-h_t^i\right)} &= f_h\left(K_t, H_t\right) \\ \frac{u_c\left(c_t^i, 1-h_t^i\right)}{u_c\left(c_{t+1}^i, 1-h_{t+1}^i\right)} &= \beta\left[f_k\left(K_{t+1}, H_{t+1}\right) + (1-\delta)\right] \end{aligned}$$

For the individual decisions we need to include the budget constraint

$$k_{t+1}^i = (1-\delta)\, k_t^i + f_h(K_t, H_t)\, h_t^i + f_k(K_t, H_t)\, k_t^i - c_t^i$$

Solving the model

- **After and only after** the individual decision rules are found, can one aggregate across individuals
  - Because the K_t and H_t are not controled by individuals
- These are found from the equilibrium conditions

$$K_t = \int_0^1 k_t^i di$$

$$H_t = \int_0^1 h_t^i di$$

- Aggregation condition for consumption

$$C_t = \int_0^1 c_t^i di$$

Solving the model

- Given that production is homogenous of degree 1

$$f_h(K_t, H_t)\, H_t + f_k(K_t, H_t)\, K_t = f(K_t, H_t)$$

- The conditions for equilibrium are

$$\begin{aligned} \frac{u_h(C_t, 1-H_t)}{u_c(C_t, 1-H_t)} &= f_h(K_t, H_t) \\ \frac{u_c(C_t, 1-H_t)}{u_c(C_{t+1}, 1-H_{t+1})} &= \beta\left[f_k(K_{t+1}, H_{t+1}) + (1-\delta)\right] \end{aligned}$$

  and the budget constraint is

$$K_{t+1} = (1-\delta)\, K_t + f(K_t, H_t) - C_t$$

The stationary states

- Solving for the stationary states, one gets

$$\begin{aligned}
\frac{u_h\left(\overline{C}, 1-\overline{H}\right)}{u_c\left(\overline{C}, 1-\overline{H}\right)} &= f_h\left(\overline{K}, \overline{H}\right) \\
\frac{1}{\beta} - (1-\delta) &= f_k\left(\overline{K}, \overline{H}\right)
\end{aligned}$$

and the budget constraint is

$$f\left(\overline{K}, \overline{H}\right) - \delta \overline{K} = \overline{C}$$

.The stationary states

- Or, the two equations in H and K of

$$\begin{aligned}
\frac{u_h\left(f\left(\overline{K}, \overline{H}\right) - \delta \overline{K}, 1-\overline{H}\right)}{u_c\left(f\left(\overline{K}, \overline{H}\right) - \delta \overline{K}, 1-\overline{H}\right)} &= f_h\left(\overline{K}, \overline{H}\right) \\
\frac{1}{\beta} - (1-\delta) &= f_k\left(\overline{K}, \overline{H}\right)
\end{aligned}$$

- These are the same two conditions one gets for the Robinson Crusoe economy (with labor decision)

.The second welfare theorem

> "The Second Fundamental Welfare Theorem. If household preferences and firm productions sets are convex, there is a complete set of markets with publicly known prices, and every agent acts as a price taker, then *any Pareto optimal outcome can be achieved as a competitive equilibrium if appropriate lump-sum transfers of wealth are arranged.*"

- What does the Second Fundamental Welfare Theorem mean?
  - A perfectly competitive economy can be solved as a social planner (one person) problem
  - This need not work if the economy is not perfectly competitive
- The above statement is for a finite dimension economy. Infinite dimensioned economies are a bit more complicated. More assumptions are needed to get competitive equilibrium. We assume a competitive equilibrium exists

.Homework

- Exercise: Find the stationary state equilibrium for an economy with

$$u\left(c_t^i, 1-h_t^i\right) = \log c_t^i + A \log\left(1-h_t^i\right),$$

$A = 1.72$, and

$$f\left(K_t, H_t\right) = \gamma K_t^{\theta} H_t^{1-\theta}$$

with $\gamma = 3$ and $\theta = .56$. Use $\beta = .98$ and $\delta = .06$. Find the numerical values of output, consumption, the capital stock, and labor supply in the stationary state using Matlab.