Resursive deterministic models

Prof. George McCandless
UCEMA

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1 Recursive deterministic models

Recursive deterministic models

- What is a recursive problem
- Nature of the problem is the same independent of the period
  1. Same maximization problem
  2. Same budget constraints
  3. Initial values can be different
- Example
  1. A household maximizes its discounted utility stream subject to a budget constraint
  2. If each period the utility **function** is the same
  3. The **form** of the budget constraints are the same
     - The values in the budget constraint can change
     - Wealth can be different in different periods

States and controls

- Three types of variables
  1. State variables
  2. Control variables
  3. Other (jump) variables

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- States variables are those predetermined at the beginning of a period

1. Capital in a Solow growth model (came from previous period)
2. Shock to technology (determined by **nature**)
3. Money stock carried over from previous period

- Control variables are those chosen to maximize some objective function
  1. Investment in the period
  2. Labor supplied in the period
  3. Consumption in the period
  4. Capital to be carried over to the next period
  5. Money holding to be carried over to the next period

- Other variables: those determined by the states and the choices for the values of the controls
  1. Output is determined by capital (normally a state) and by labor supply (normally a control)
  2. If output is determined and investment is a control, consumption is determimed from budget constraints

Policy Function

- The solution we look for is called a **Policy Function**
- A policy function gives
  - The optimizing values for the time t **Controls**
  - As a function of the values of the time t **States**
  - A policy function tells what to do based on what is happening
    * was developed in the 1950's
    * separately in the US and in the Soviet Union
    * countries needed controls for their rockets

Infinite horizon problem with states and controls

Robinson Crusoe want to maximize the discounted utility

$$\max \sum_{i=0}^{\infty} \beta^i u(c_{t+i}),$$

subject to the budget restrictions,

[5cm]

5cm

$$k_{t+1} = (1-\delta)k_t + i_t,$$

5cm

$$y_t = f(k_t) = c_t + i_t.$$

k_t is the state variable

possible choices of controls

k_{t+1} (choosing k_{t+1} determines the required i_t and, from the second budget constraint, c_t

c_t (choosing c_t determines i_t and this determines k_{t+1}

i_t (choosing i_t determines c_t from one budget constraint and k_{t+1} from the other

As the problem is written, c_t is the control

.Writing the problem with capital as the control

- Use the budget constraint to write

$$c_t = f(k_t) + (1-\delta)k_t - k_{t+1}$$

- Substitute this into the utility function to get

$$\max \sum_{i=0}^{\infty} \beta^i u \left[f(k_{t+i}) + (1-\delta)k_{t+i} - k_{t+i+1} \right]$$

- In this case, k_{t+i+1} is a control in period $t+i$
  - k_{t+i} is the state in that period

.The value function

Definition 1 *For given values of the state variables at time t, the value function gives the value of the discounted objective function when that objective function is being maximized.*

The value function is a function of the **state variables**

The discounted objective function is being optimized

Example:

$$V(k_t) = \max_{\{k_s\}_{s=t+1}^{\infty}} \sum_{i=0}^{\infty} \beta^i u(f(k_{t+i}) - k_{t+1+i} + (1-\delta)k_{t+i})$$

.The value function

- Let x_t be the state variables.
- $V(x_t)$ is the value of

$$\sum_{i=0}^{\infty} \beta^i u(f(k_{t+i}) - k_{t+1+i} + (1-\delta)k_{t+i})$$

- When the sequence of $\{k_s\}_{s=t+1}^{\infty}$ has been chosen to maximize it
- $V(x_t)$ is a function of the state variables and its value changes when the value of the state variables change (in this case, k_0)

Recursive problems

- The time t problem

$$V(k_t) = \max_{\{k_s\}_{s=t+1}^{\infty}} \sum_{i=0}^{\infty} \beta^i u(f(k_{t+i}) - k_{t+1+i} + (1-\delta)k_{t+i})$$

  is recursive

- In time $t+1$, Robinson Crusoe is solving

$$V(k_{t+1}) = \max_{\{k_s\}_{s=t+2}^{\infty}} \sum_{i=0}^{\infty} \beta^i u(f(k_{t+1+i}) - k_{t+2+i} + (1-\delta)k_{t+1+i})$$

Decomposing the time t problem
The time t problem

$$V(k_t) = \max_{\{k_s\}_{s=t+1}^{\infty}} \sum_{i=0}^{\infty} \beta^i u(f(k_{t+i}) - k_{t+1+i} + (1-\delta)k_{t+i})$$

can be written as

$$\begin{aligned} V(k_t) &= \max_{k_{t+1}} \Big[u(f(k_t) - k_{t+1} + (1-\delta)k_t) \\ &\quad + \beta \max_{\{k_s\}_{s=t+2}^{\infty}} \sum_{i=0}^{\infty} \beta^i u(f(k_{t+1+i}) - k_{t+2+i} + (1-\delta)k_{t+1+i}) \Big] \end{aligned}$$

or as

$$V(k_t) = \max_{k_{t+1}} [u(f(k_t) - k_{t+1} + (1-\delta)k_t) + \beta V(k_{t+1})]$$

The Bellman equation

- The recursive equation

$$V(k_t) = \max_{k_{t+1}} [u(f(k_t) - k_{t+1} + (1-\delta)k_t) + \beta V(k_{t+1})]$$

  is called a Bellman equation

- It is recursive because $V(k_t)$ dependes on the value of the same function $V(\cdot)$ but evaluated at k_{t+1}
- It is a one period problem

  - One only chooses the value of k_{t+1}
  - Notice that all future utility is captured in $V(k_{t+1})$
  - Choice of k_{t+1} will change the value of $V(k_{t+1})$

- Lots of systems work this way
  - riding a bicycle, driving a car, flying a glider

First order conditions

- Take the derivative of $V(k_t)$ with respect to k_{t+1}
- Get
$$0 = -u'(f(k_t) - k_{t+1} + (1-\delta)k_t) + \beta V'(k_{t+1})$$
- Problem is that we do not know the function $V(k_{t+1})$ nor its derivative $V'(k_{t+1})$

Benveniste - Scheinkman envelope theorem conditions

- Benveniste - Scheinkman give conditions under which one can find $V'(\cdot)$
- Take derivative of
$$V(k_t) = \max_{k_{t+1}} [u(f(k_t) - k_{t+1} + (1-\delta)k_t) + \beta V(k_{t+1})]$$
with respect to k_t
- Get
$$V'(k_t) = u'(f(k_t) - k_{t+1} + (1-\delta)k_t)\,(f'(k_t) + (1-\delta))$$
which we can evaluate at k_{t+1}
- The result is called an envelope theorem

First order and B-S envelope conditions

- Combine first order and envelope conditions
- Get the Euler equation
$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta\,(f'(k_{t+1}) + (1-\delta))\,.$$
- In a stationary state, where $c_t = c_{t+1}$, this is
$$\frac{1}{\beta} - (1-\delta) = f'(\bar{k}).$$

.General version of problem

Let x_t be the state variables and y_t the controls

We want solve

$$V(x_t) = \max_{\{y_s\}_{s=t}^{\infty}} \sum_{s=t}^{\infty} \beta^{s-t} F(x_s, y_s)$$

subject to the set of budget constraints

$$x_{s+1} = G(x_s, y_s).$$

The functions, $F(\cdot,\cdot)$ and $G(\cdot,\cdot)$, are the same for all periods

Both time t state variables and control variables can be in the objective function and the budget constraints at time t.

This can be written as a Bellman equation,

$$V(x_t) = \max_{y_t} \left[F(x_t, y_t) + \beta V(x_{t+1}) \right],$$

subject to the budget constraints

$$x_{t+1} = G(x_t, y_t),$$

.General version of problem

The Bellman equation can be written as

$$V(x_t) = \max_{y_t} \left[F(x_t, y_t) + \beta V(G(x_t, y_t)) \right]$$

We solve for a **policy function** of the form

$$y_t = H(x_t)$$

The time t controls are functions of the time t state variables

Notice that the problem is a **functional equation** and that the solution is the **function** $y_t = H(x_t)$

.General version of problem: the first order conditions

- Taking the derivative of the Bellman equation gives

$$0 = F_y(x_t, y_t) + \beta V'(G(x_t, y_t)) G_y(x_t, y_t)$$

- As before we can find the Benveniste-Scheinkman envelope theorem

$$V'(x_t) = F_x(x_t, y_t) + \beta V'(G(x_t, y_t)) G_x(x_t, y_t)$$

  - If $G_x(x_t, y_t) = 0$
  - The envelope condition is simply $V'(x_t) = F_x(x_t, y_t)$

- The solution can be written as the Euler equation

$$0 = F_y(x_t, y_t) + \beta F_x(G(x_t, y_t), y_{t+1}) G_y(x_t, y_t)$$

- If the function, $F_x(G(x_t, y_t), y_{t+1})$, is independent of y_{t+1},
- This equation can be solved for, $y_t = H(x_t)$
- Normally, explicit solutions cannot be found

Conditions for the envelope theorem (from Benveniste-Scheinkman)

- Conditions are (for our form of the model)
  - $x_t \in X$ where X is convex and with non-empty interior
  - $y_t \in Y$ where Y is convex and with non-empty interior
  - $F(x_t, y_t)$ is continuous and differentiable
  - $G(x_t, y_t)$ is continuous and differentiable and invertible in y_t
- This gives enough structure so the envelope theorem holds

Newer, more general results in Milgrom and Segel
Approximation of the value function

What happens if $G_x(x_t, y_t) \neq 0$?

One can approximate the value function numerically

Great contribution of Bellman

Choose some initial **function** $V_0(x_t)$

Most any function will do

a good one is $V_0(x_t) = c$

where c is a constant (0, for example)

Find (approximately) the **function** $V_1(x_t)$

$$V_1(x_t) = \max_{y_t} \left[F(x_t, y_t) + \beta V_0(G(x_t, y_t)) \right]$$

over a dense set of values from the domain of x_t

One now has the function $V_1(x_t)$
Approximation of the value function (continued)

- Using this function $V_1(x_t)$, find

$$V_2(x_t) = \max_{y_t} \left[F(x_t, y_t) + \beta V_1(G(x_t, y_t)) \right]$$

  over a dense set of values from the domain of x_t

  - one will need to interpolate the function $V_1(x_t)$

- when the needed $G(x_t, y_t)$ is not part of the dense set of x_t
- linear interpolation is normally good enough

- Using $V_2(x_t)$ repreat the process
- Get a sequence $\{V_i(x_t)\}_{i=0}^{\infty}$
- Bellman showed that $\{V_i(x_t)\}_{i=0}^{\infty} \longrightarrow V(x_t)$
- Once you have $V(x_t)$ finding $y_t = H(x_t)$ is easy
  - Actually, one finds a sequence $\{H_i(x_t)\}_{i=0}^{\infty} \longrightarrow H(x_t)$
  - while finding $\{V_i(x_t)\}_{i=0}^{\infty} \longrightarrow V(x_t)$
- Why does this work? Answer $= \beta$

.Problems of dimensionality
How well do we choose to approximate the function
How many points in the domain of x_t
If $x_t \in \mathbb{R}^1$ we can choose lots of points, M points
As dimensionality of x_t grows (say to $\mathbb{R}^N$)
number of points needed is M^N which can be very large
.Comparing example economy to general problem 1: using B-S

The objective function is

$$F(x_t, y_t) = u(f(k_t) - k_{t+1} + (1-\delta)k_t)$$

The budget constraint is

$$k_{t+1} = x_{t+1} = G(x_t, y_t) = y_t = k_{t+1}$$

or

$$k_{t+1} = k_{t+1}$$

The first order condition is

$$\begin{aligned} 0 &= F_y(x_t, y_t) + \beta V'(G(x_t, y_t))G_y(x_t, y_t) \\ &= -u'(f(k_t) - k_{t+1} + (1-\delta)k_t) + \beta V'(G(x_t, y_t)) \cdot 1 \end{aligned}$$

Because $\partial k_{t+1}/\partial k_t = 0$, the B-S envelope theorem condition is

$$V'(x_t) = F_x(x_t, y_t) = u'(f(k_t) - k_{t+1} + (1-\delta)k_t)\,(f'(k_t) + (1-\delta))$$

.Comparing example economy to general problem 1: using B-S

- Use this $V'(\cdot)$ in the first order conditions to get the Euler equation

$$\begin{aligned} 0 = & -u'(f(k_t) - k_{t+1} + (1-\delta)k_t) \\ & +\beta \left[u'(f(k_{t+1}) - k_{t+2} + (1-\delta)k_{t+1})\,(f'(k_{t+1}) + (1-\delta))\right]. \end{aligned}$$

- we can find the stationary state where $k_t = k_{t+1} = k_{t+2} = \overline{k}$ as

$$f'(\overline{k}) = \frac{1}{\beta} - (1-\delta)$$

.Comparing example economy to general problem 2

- We can solve the problem a different way
- Let the objective function be

$$F(x_t, y_t) = u(c_t)$$

- The budget constraint is

$$k_{t+1} = x_{t+1} = G(x_t, y_t) = f(k_t) + (1-\delta)k_t - c_t$$

- The Bellman equation is

$$V(k_t) = \max_{c_t} \left[u(c_t) + \beta V\left(f(k_t) + (1-\delta)k_t - c_t\right)\right]$$

  - Notice that the budget constraint is already in $V(k_{t+1})$

- The derivative of the budget constraint is

$$\frac{\partial G(x_t, y_t)}{\partial x_t} = f'(k_t) + (1-\delta) \neq 0$$

  - Can't use B-S method

.Approximation of the Value function

To approximate the value function need explicit functions for $u(c_t)$ and $f(k_t)$

Let $f(k_t) = k_t^{\theta}$ and $u(c_t) = \ln(c_t)$

Let $\delta = .1$, $\theta = .36$, and $\beta = .98$ (consistent with annual data for US)

The Bellman equation is

$$V(k_t) = \max_{k_{t+1}} \left[\ln(k_t^{\theta} - k_{t+1} + (1-\delta)k_t) + \beta V(k_{t+1})\right]$$

Note: stationary state $\overline{k} = 5.537$ (how do you find this?)

.Approximation of the Value function

- Choose $V_0(\cdot) = 0$ (a constant initial guess for value function)
- Find $V_1(\cdot)$ using

$$\begin{aligned} V_1(k_t) &= \max_{k_{t+1}} \left[\ln(k_t^{\theta} - k_{t+1} + (1-\delta)k_t) + \beta V_0(k_{t+1})\right] \\ &= \max_{k_{t+1}} \left[\ln(k_t^{.36} - k_{t+1} + .9k_t) + .98 \cdot 0\right] \end{aligned}$$

  for a dense set of k_t

- Find $V_2(\cdot)$ using

$$V_2(k_t) = \max_{k_{t+1}} \left[\ln(k_t^{.36} - k_{t+1} + .9k_t) + .98 \cdot V_1(k_{t+1}) \right]$$

  for a dense set of k_t. Use linear interpolation of $V_1(k_{t+1})$ between known points

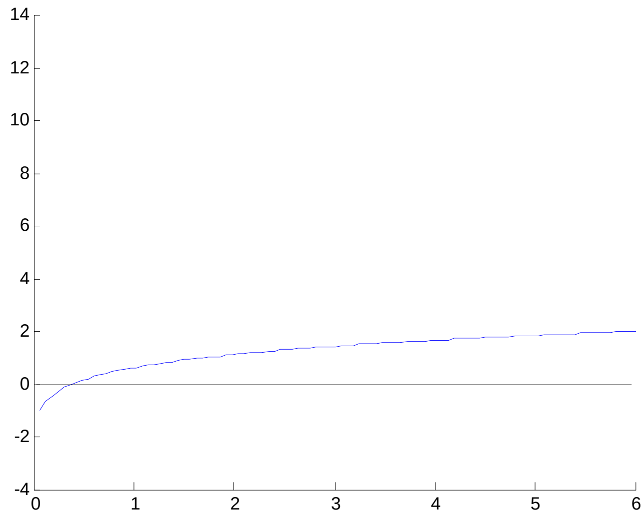
- Repeat N times. Get approximate $V(k_t)$ function (as close as you want)

.Computer program
Main program

```
global vlast beta delta theta k0 kt
hold off
hold all
%set initial conditions
vlast=zeros(1,100);
k0=0.06:0.06:6;
beta=.98;
delta=.1;
theta=.36;
numits=240;
%begin the recursive calculations
for k=1:numits
     for j=1:100
         kt=j*.06;
         %find the maximum of the value function
         ktp1=fminbnd(@valfun,0.01,6.2);
         v(j)=-valfun(ktp1);
         kt1(j)=ktp1;
     end
 if k/48==round(k/48)
     %plot the steps in finding the value function
     plot(k0,v)
     drawnow
 end
 vlast=v;
end
hold off
% plot the final policy function
plot(k0,kt1)
```

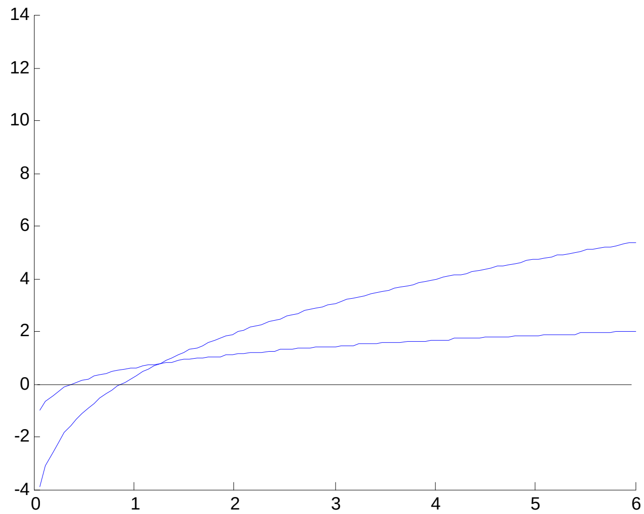
.Computer program
Subroutine (valfun.m) to calculate value function

```
function val=valfun(k)
global vlast beta delta theta k0 kt
%smooth out the previous value function
g=interp1(k0,vlast,k,'linear');
```




```
%Calculate consumption with given parameters
kk=kt^theta-k+(1-delta)*kt;
if kk <= 0
     %to keep values from going negative
     val=-888-800*abs(kk);
else
     %calculate the value of the value function at k
     val=log(kk)+beta*g;
end
%change value to negative since "fminbnd" finds minimum
val=-val;
```

.$V(k_t)$ after one iteration
.$V(k_t)$ after ten iterations
.$V(k_t)$ after 50 iterations
.$V(k_t)$ after 100 iterations
.$V(k_t)$ after 200 iterations
.The policy function after 200 interatioins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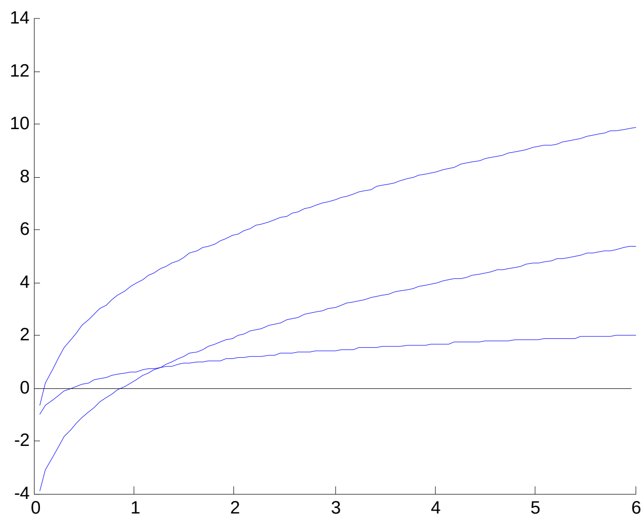
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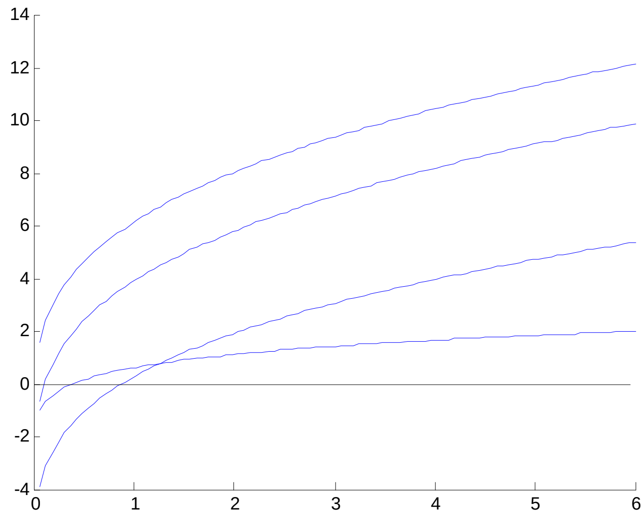
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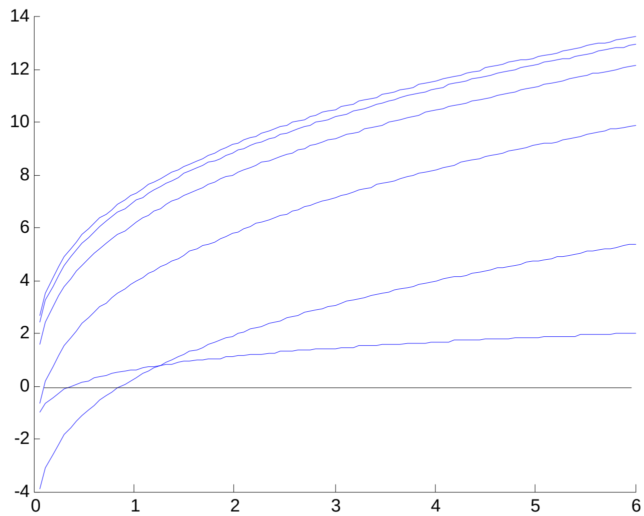
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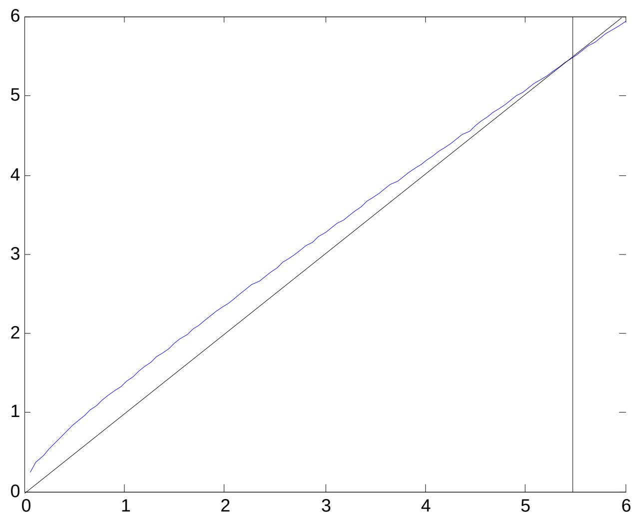

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