Stochastic Recursive Models

George McCandless
UCEMA

Spring 2007

1 Stochastic Recursive Models

Probability

- A probability space is a triplet:

$$(\Omega, \mathcal{F}, P)$$

- Where
  - Ω is a set of all states of nature that can occur
  - $\mathcal{F}$ is a collection of subsets of Ω, each subset is an "event"
  - P is a probability measure over $\mathcal{F}$

Probability for finite states of nature

- Suppose that Ω is a finite set
- We can sometimes impose the probability measure directly on Ω
  - if each state of nature can be considered an event
- Define p_i = probability that event A_i will occur
- If the probabilities are independent
  - $p(A_i, A_j) = p_i + p_j$
- We can have non-independent events
  - $E_1 = (A_1, A_2)$
  - $E_2 = (A_2, A_3)$
  - Then $p(E_1 \cup E_2) = p_1 + p_2 + p_3 < p(E_1) + p(E_2)$

Probability for finite states of nature

- Let $\Omega = \{A_1, A_2\}$
- Then the largest possible $\mathcal{F}$ is
  - $[]$, $[A_1]$, $[A_2]$, $[A_1, A_2]$
- A possible probability measure is
  - $p([]) = 0$, $p([A_1]) = \overline{p}$, $p([A_2]) = 1 - \overline{p}$, and $p([A_1, A_2]) = 1$
- Another possible is
  - $p([]) = \widehat{p}_1$, $p([A_1]) = \overline{p}$, $p([A_2]) = 1 - \widehat{p}_1 - \overline{p}$, and $p([A_1, A_2]) = 1 - \widehat{p}_1$

Probability for continuous states of nature

- Let $\Omega = [0, 300]$, all the points on the line from 0 to 300 (inclusive)
- Let $\mathcal{F}$ be all measureable sets of Ω
- P assigns probabilities to these measurable sets
- $p([45, 46.1])$ is the probability of the value falling between 45 and 46.1
- $p(\pi, 4)$ is the probability of the value falling between π and 4
- In general, $p(x) = 0$, where x is a specific number
  - Not always the case
    - $p(x)$, where x is the daily rainfall in Buenos Aries
    - $p(0)$ is the probability it won't rain
    - $p(0) > 0$

A simple stochastic model

- Robinson Crusoe model with stochastic technology
- Production function
$$y_t = A^t f(k_t),$$
with the technology A^t determined by
$$A^t = \begin{cases} A_1 \text{ with probability } p_1 \\ A_2 \text{ with probability } p_2 \end{cases}$$
and we assume that $A_1 > A_2$ and that $p_2 = 1 - p_1$
- RC's utility function is
$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$
subject to the budget constraint
$$k_{t+1} = A^t f(k_t) + (1 - \delta) k_t - c_t.$$

- This is an **expected** utility function: that is what E_0 means

Stochastic Value function

- The value of discounted expected utility at time 0 when the observed technology shock is A_1 is

$$V(k_0, A_1) = \max_{\{c_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

  subject to the budget constraint for $t = 0$,

$$k_1 = A_1 f(k_0) + (1 - \delta)k_0 - c_0,$$

  and those for $t \geq 1$,

$$k_{t+1} = A^t f(k_t) + (1 - \delta)k_t - c_t,$$

  and the independent realizations of $A^t = [A_1, A_2]$ with probabilities $[p_1, p_2]$

- There is a similar expression for $V(k_0, A_2)$

Stochastic Value function: recursive format

- The value function is

$$V(k_0, A^0) = \max_{c_0} u(c_0) + \beta E_0 V\left(k_1, A^1\right)$$

  subject to the budget constraint

$$k_1 = A^0 f(k_0) + (1 - \delta)k_0 - c_0.$$

- Here we are taking c_0 as the control
- The states are k_0 and A^0
- Notice the expectations operator
  - In the second part of the value function
  - This is because we do not know the realization of A^1

Stochastic Value function: with k_{t+1} as control

- The value function is

$$V(k_t, A^t) = \max_{k_{t+1}} u(A^t f(k_t) + (1 - \delta)k_t - k_{t+1}) + \beta E_t V\left(k_{t+1}, A^{t+1}\right)$$

  and the budget constraint is

$$k_{t+1} = G(x_t, y_t) = k_{t+1}$$

- The control at time t is the state at time $t+1$

We solve for a **plan**, a function such that

$$k_{t+1} = H(k_t, A^t)$$

A plan solves (without maximization)

$$\begin{aligned} V(k_t, A^t) &= u(A^t f(k_t) + (1-\delta)k_t - H(k_t, A^t)) \\ &\quad + \beta E_t V\left(H(k_t, A^t), A^{t+1}\right) \end{aligned}$$

.General version of the problem

- Write the value function as

$$V(x_t, z_t) = \max_{\{y_s\}_{s=t}^{\infty}} E_t \sum_{s=t}^{\infty} \beta^{s-t} F(x_s, y_s, z_s),$$

  subject to

$$x_{s+1} = G(x_s, y_s, z_s) \text{ for } s \geq t$$

- x_t is the set of "regular" state variables
- z_t is the set of state variables determined by nature
  - these are the stochastic state variables.
- y_t are the control variables
- Both $F(x_s, y_s, z_s)$ and $G(x_s, y_s, z_s)$ can contain the stochastic state variables.

.General version of the problem

The recrusive version of this problem is

$$V(x_t, z_t) = \max_{y_t} \left[F(x_t, y_t, z_t) + \beta E_t V(x_{t+1}, z_{t+1})\right]$$

subject to

$$x_{t+1} = G(x_t, y_t, z_t)$$

A solution is a plan,

$$y_t = H(x_t, z_t)$$

where

$$\begin{aligned} V(x_t, z_t) &= F(x_t, H(x_t, z_t), z_t) \\ &\quad + \beta E_t V(G(x_t, H(x_t, z_t), z_t), z_{t+1}) \end{aligned}$$

.General version of the problem: first order conditions

- The first order conditions are

$$0 = F_y(x_t, y_t, z_t) + \beta E_t \left[V_x(G(x_t, y_t, z_t), z_{t+1}) G_y(x_t, y_t, z_t)\right]$$

The Benveniste-Scheinkman envelope theorem condition is

$$V_x(x_t, z_t) = F_x(x_t, y_t, z_t) + \beta E_t \left[V_x(G(x_t, y_t, z_t), z_{t+1}) G_x(x_t, y_t, z_t)\right]$$

If we can choose the controls so that $G_x(x_t, y_t, z_t) = 0$, this becomes

$$V_x(x_t, z_t) = F_x(x_t, y_t, z_t)$$

One can write the **stochastic Euler equation** as

$$0 = F_y(x_t, y_t, z_t) + \beta E_t \left[F_x(G(x_t, y_t, z_t), y_{t+1}, z_{t+1}) G_y(x_t, y_t, z_t)\right]$$

Solving for the value function

- One can find an approximation of the value function from

$$V_{j+1}(x_t, z_t) = \max_{y_t} \left[F(x_t, y_t, z_t) + \beta E_t V_j(G(x_t, y_t, z_t), z_{t+1})\right]$$

- Beginning with some function $V_0(\cdot)$ (usually a constant)
- Need to solve over dense set of $X \times Z$
  - where X is the domain of the state variables, x_t
  - Z is the domain of the states of nature, z_t

Problem of dimensionality

- Problem of dimensionality is worse than in the deterministic case
- In the deterministic case is based on
  - the number of state variables
  - the size of the dense subset of each state variable we use
- In the stochastic state
  - these two problems continue
  - add
    - the dimension of the shocks (if finite)
    - the dense subset of the shocks (if continuous)

Finding the value function for our simple economy

- In the growth economy, the technology level can be $[A_1, A_2]$
- This gives two values functions of the form

$$\begin{aligned} V(k_t, A_1) &= \max_{k_{t+1}} u(A_1 f(k_t) + (1-\delta)k_t - k_{t+1}) \\ &+\beta \left[p_1 V\left(k_{t+1}, A_1\right) + p_2 V\left(k_{t+1}, A_2\right)\right] \end{aligned}$$

and

$$\begin{aligned} V(k_t, A_2) &= \max_{k_{t+1}} u(A_2 f(k_t) + (1-\delta)k_t - k_{t+1}) \\ &+\beta \left[p_1 V\left(k_{t+1}, A_1\right) + p_2 V\left(k_{t+1}, A_2\right)\right] \end{aligned}$$

- Notice how the probabilities enter
- We need to find two functions, $V(\cdot, A_1)$ and $V(\cdot, A_2)$

.The recursive approximation

- Same recursive approximation as before
- Difference is that we need to find two equations at each iteration
- Given $V_0(\cdot, A_1)$ and $V_0(\cdot, A_2)$, we find

$$\begin{aligned} V_1(k_t, A_1) &= \max_{k_{t+1}} u(A_1 f(k_t) + (1-\delta)k_t - k_{t+1}) \\ &+\beta \left[p_1 V_0\left(k_{t+1}, A_1\right) + p_2 V_0\left(k_{t+1}, A_2\right)\right], \end{aligned}$$

and

$$\begin{aligned} V_1(k_t, A_2) &= \max_{k_{t+1}} u(A_2 f(k_t) + (1-\delta)k_t - k_{t+1}) \\ &+\beta \left[p_1 V_0\left(k_{t+1}, A_1\right) + p_2 V_0\left(k_{t+1}, A_2\right)\right], \end{aligned}$$

- Repeat, finding $V_N(\cdot, A_1)$ and $V_N(\cdot, A_2)$ until sufficiently close

.Example

- Used $\delta = .1$, $\beta = .98$, $A_1 = 1.75$, $p_1 = .8$, $A_2 = .75$,
- and $p_2 = .2$, $V_0(\cdot, A_1) = 20$ and $V_0(\cdot, A_2) = 20$
- Graph of iterations

.The two policy functions (the plans)
.A simulation of the economy
.What do I do if my probability space is large?

- Suppose the space is $\{A_1, A_2, ..., A_N\}$ with probabilities $\{p_1, p_2, ..., p_N\}$

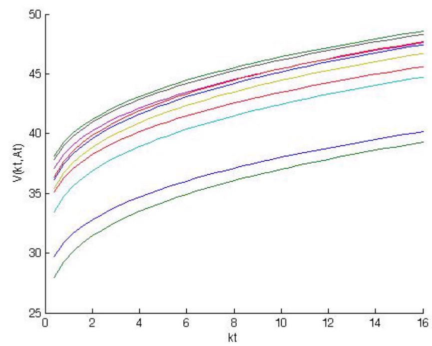


Figure 1: Iterations on the value function

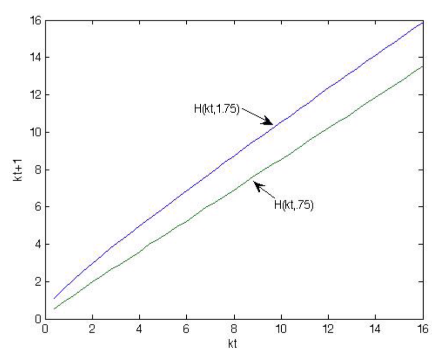


Figure 2: The plans

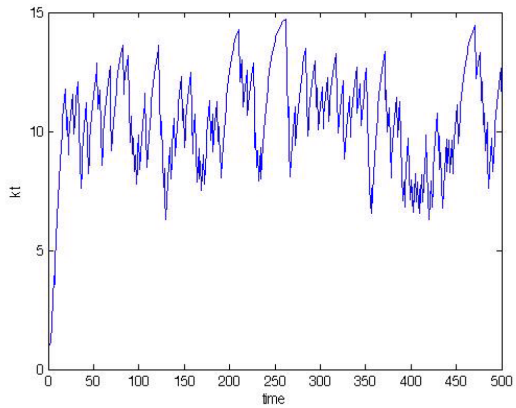


Figure 3: A simulated time path

- I just need to have N value functions and continue as above
- What do I do if my probability space is continuous?
  - Take a dense finite subset of the probability space
  - This is OK since nature chooses the draw from the probability space
  - Don't have the interpelation problems of a choice variable

Markov chains

- The above simulation shows little persistence
- Markov chains are a way of adding persistence to the shocks
  - Note that the presistence is in the stochastic part
  - The economic model is not generating this persistence
- Structure of a Markov chain
  - The probabilities at time t of the time $t+1$ states of nature
  - depend on the state of nature at time t
- Consider our example with two states of nature $[A_1, A_2]$
- Let the probabilities be

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

where p_{ij} is the probability of going to state j given you are in state i

Probabilities in Markov chains

- These are **conditional probabilities**
- If one is in state of nature 1 at time t
  - The probabilities for time $t+1$ are

$$\begin{bmatrix} p_{11} & p_{12} \end{bmatrix}$$

- If one is in state of nature 2 at time t
  - The probabilities for time $t+1$ are

$$\begin{bmatrix} p_{21} & p_{22} \end{bmatrix}$$

- Example

$$P = \begin{bmatrix} .97 & .03 \\ .1 & .9 \end{bmatrix}$$

Unconditional probabilites

- What is the probability that one will be in state of nature j at some far distant date
- Does this depend on the current state of nature
- Given the state at time 0, the distributiion for period 1 is $p_0 = \begin{bmatrix} p_{01} & p_{02} \end{bmatrix}$
- Then the distribution for period 2 is

$$p_0 P = \begin{bmatrix} p_{01} & p_{02} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

- The distribution for period 3 is

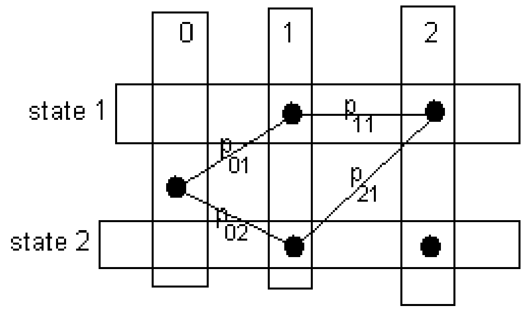
$$p_0 P P = \begin{bmatrix} p_{01} & p_{02} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

- The distribution for period N is

$$p_0 P^{N-1}$$

Tree diagrams of the probabilities
Converge to an unconditional probability

- What happens as N gets large
- Use our example probability matrix (leave p_0 out for the moment)



- Start with

$$P = \begin{bmatrix} .97 & .03 \\ .1 & .9 \end{bmatrix}$$

- $PP = P^2$ is

-

$$P^2 = \begin{bmatrix} .97 & .03 \\ .1 & .9 \end{bmatrix} \begin{bmatrix} .97 & .03 \\ .1 & .9 \end{bmatrix} = \begin{bmatrix} 0.9439 & 0.0561 \\ 0.1870 & 0.8130 \end{bmatrix}$$

.Using a doubling algorythm

$$P^4 = P^2P^2 = \begin{bmatrix} 0.944 & 0.056 \\ 0.187 & 0.813 \end{bmatrix} \begin{bmatrix} 0.944 & 0.056 \\ 0.187 & 0.813 \end{bmatrix} = \begin{bmatrix} 0.901 & 0.099 \\ 0.328 & 0.672 \end{bmatrix}$$

$$P^8 = P^4P^4 = \begin{bmatrix} 0.8450 & 0.1550 \\ 0.5168 & 0.4832 \end{bmatrix}$$

$$P^{16} = P^8P^8 = \begin{bmatrix} 0.7941 & 0.2059 \\ 0.6864 & 0.3136 \end{bmatrix}$$

$$P^{32} = P^{16}P^{16} = \begin{bmatrix} 0.7719 & 0.2281 \\ 0.7603 & 0.2397 \end{bmatrix}$$

$$P^{64} = P^{32}P^{32} = \begin{bmatrix} 0.7693 & 0.2307 \\ 0.7691 & 0.2309 \end{bmatrix}$$

$$P^{128} = P^{64}P^{64} = \begin{bmatrix} 0.7692 & 0.2308 \\ 0.7692 & 0.2308 \end{bmatrix}$$

.Why the initial distribution does not matter

- Notice the rows of P^{128},

$$P^{128} = P^{64}P^{64} = \begin{bmatrix} 0.7692 & 0.2308 \\ 0.7692 & 0.2308 \end{bmatrix}$$

- They are identical
- Let $p_0 = \left[\begin{array}{cc} p_{01} & p_{02} \end{array} \right]$
- Then

$$\begin{aligned} p_0 P^{128} &= \left[\begin{array}{cc} p_{01} & p_{02} \end{array} \right] \left[\begin{array}{cc} 0.7692 & 0.2308 \\ 0.7692 & 0.2308 \end{array} \right] \\ &= \left[\begin{array}{cc} 0.7692 & 0.2308 \end{array} \right] \end{aligned}$$

- Initial distribution does not matter in the long run
  - for the unconditional distribution

.Value functions with markov chains

- The value functions for our economy can be written as

$$\begin{aligned} V(k_t, A_1) &= \max_{k_{t+1}} [u(A_1 f(k_t) + (1-\delta)k_t - k_{t+1}) \\ &\quad + \beta \left[p_{11} V\left(k_{t+1}, A_1\right) + p_{12} V\left(k_{t+1}, A_2\right) \right]], \end{aligned}$$

  and

$$\begin{aligned} V(k_t, A_2) &= \max_{k_{t+1}} [u(A_2 f(k_t) + (1-\delta)k_t - k_{t+1}) \\ &\quad + \beta \left[p_{21} V\left(k_{t+1}, A_1\right) + p_{22} V\left(k_{t+1}, A_2\right) \right]], \end{aligned}$$

- Note the probabilities in each equation
- These can be solved recursively
  - beginning with some $V_0(\cdot, A_1)$ and $V_0(\cdot, A_2)$
  - just need to keep track of which probabilities to use

.Example economy
[5cm]
5cm

- We use a markov chain of

$$P = \left[\begin{array}{cc} .9 & .1 \\ .4 & .6 \end{array} \right]$$

6cm

- Unconditional probabilities

$$P^{\infty} = \left[\begin{array}{cc} .8 & .2 \\ .8 & .2 \end{array} \right]$$

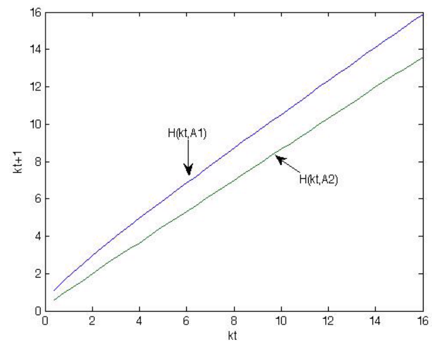


Figure 4: The plans with Markov chains

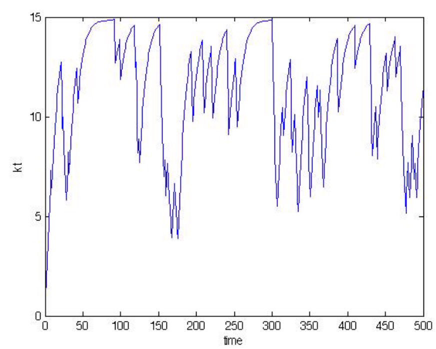


Figure 5: A simulation with Markov chains

- Get the plans of (similar but not identical to the last problem)

Simulated economy with markov chain (same shocks as in other)

- Note the increased presistence

Computer program for Markov chains

```
global vlast1 vlast2 beta delta theta k0 kt At p1 p2
hold off
hold all
vlast1=20*ones(1,40);
vlast2=vlast1;
k0=0.4:0.4:16;
kt11=k0;
kt12=k0;
beta=.98;
delta=.1;
theta=.36;
A1=1.75;
p11=.9;
p12=1-p11;
p21=.4;
p22=1-p21;
A2=.75;
numits=250;
for k=1:numits
     for j=1:40
         kt=k0(j);
         At=A1;
        p1=p11;
         p2=p12;
         z=fminbnd(@valfunsto,.41,15.99);
        v1(j)=-valfunsto(z);
        kt11(j)=z;
         At=A2;
         p1=p21;
         p2=p22;
        z=fminbnd(@valfunsto,.41,15.99);
         v2(j)=-valfunsto(z);
         kt12(j)=z;
     end
     if k/50==round(k/50)
         plot(k0,v1,k0,v2)
         drawnow
     end
     vlast1=v1;
```

```
     vlast2=v2;
end
hold off
%plot(k0,kt11,k0,kt12)
```

Subroutine valfunsto

Note that interpolation of the previous value function is linear.

```
function val=valfunsto2(x)
global vlast1 vlast2 beta delta theta k0 kt At p1 p2
k=x;
g1=interp1(k0,vlast1,k,'linear');
g2=interp1(k0,vlast2,k,'linear');
kk=At*kt^theta-k+(1-delta)*kt;
val=log(kk)+beta*(p1*g1+p2*g2);
val=-val;
```