Hansen's basic RBC model

George McCandless
UCEMA

Spring 2007

1 Hansen's RBC model

Hansen's RBC model

- First RBC model was Kydland and Prescott
  - (1982) "Time to build and aggregate fluctuations," Econometrica
  - Complicated
    - lagged cumulative investment
    - strange utility function
    - Lots added to look for presistence
- Hansen's model much simpler
  - (1985) "Indivisible labor and the business cycle," Journal of Monetary Economics
  - Simple
  - Added indivisible labor to gain persistence and covariance with output
  - Set rules for RBC game
    - Match second moments
    - Newer rule: match impulse response functions

Hansen's basic model

- Robinson Crusoe maximizes the discounted utility function

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

- The specific utility functions

$$u(c_t, 1 - h_t) = \ln c_t + A \ln(1 - h_t)$$

with $A > 0$.

- The production function is

$$f(\lambda_t, k_t, h_t) = \lambda_t k_t^\theta h_t^{1-\theta}$$

- λ_t is a random technology variable that follows the process

$$\lambda_{t+1} = \gamma\lambda_t + \varepsilon_{t+1}$$

for $0 < \gamma < 1$. ε_t iid, positive, bounded above, $E\varepsilon_t = 1 - \gamma$.

  - $\Longrightarrow E\ \lambda_t$ is 1 and $\lambda_{t+1} > 0$.

Hansen's basic model (continued)

- Capital accumulation follows the process

$$k_{t+1} = (1 - \delta)k_t + i_t$$

- The feasibility constraint is

$$f(\lambda_t, k_t, h_t) \geq c_t + i_t$$

Bellmans equation

The basic Bellmans equation

$$V(k_t, \lambda_t) = \max_{c_t, h_t} \left[\ln c_t + A \ln(1 - h_t) + \beta E_t \left[V(k_{t+1}, \lambda_{t+1}) \mid \lambda_t\right]\right]$$

subject to

$$\begin{aligned} \lambda_t k_t^\theta h_t^{1-\theta} &\geq c_t + i_t, \\ \lambda_{t+1} &= \gamma\lambda_t + \varepsilon_{t+1}, \text{ and} \\ k_{t+1} &= (1 - \delta)k_t + i_t. \end{aligned}$$

Simpler to write as

$$\begin{aligned} V(k_t, \lambda_t) = \max_{k_{t+1}, h_t} &\left[\ln\left(\lambda_t k_t^\theta h_t^{1-\theta} + (1 - \delta)k_t - k_{t+1}\right)\right. \\ &\left. + A \ln(1 - h_t) + \beta E_t \left[V(k_{t+1}, \lambda_{t+1}) \mid \lambda_t\right]\right] \end{aligned}$$

k_{t+1} and h_t are control variables

First order conditions

- First order conditions are

$$\frac{\partial V(k_t,\lambda_t)}{\partial k_{t+1}} = 0 = -\frac{1}{\lambda_t k_t^\theta h_t^{1-\theta} + (1-\delta)k_t - k_{t+1}} + \beta E_t \left[V_k(k_{t+1},\lambda_{t+1}) \mid \lambda_t \right]$$

and

$$\frac{\partial V(k_t,\lambda_t)}{\partial h_t} = 0 = (1-\theta)\frac{1}{\lambda_t k_t^\theta h_t^{1-\theta} + (1-\delta)k_t - k_{t+1}}\left(\lambda_t k_t^\theta h_t^{-\theta}\right) - A\frac{1}{1-h_t}$$

- The Benveniste-Scheinkman envelope theorem condition is

$$\frac{\partial V(k_t,\lambda_t)}{\partial k_t} = \frac{1}{\lambda_t k_t^\theta h_t^{1-\theta} + (1-\delta)k_t - k_{t+1}}\left(\theta\lambda_t k_t^{\theta-1} h_t^{1-\theta} + (1-\delta)\right)$$

Simplifying the first order conditions

- First order conditions can be written as

$$\frac{1}{\lambda_t k_t^\theta h_t^{1-\theta} + (1-\delta)k_t - k_{t+1}} = \beta E_t\left[\frac{\theta\lambda_{t+1}k_{t+1}^{\theta-1}h_{t+1}^{1-\theta} + (1-\delta)}{\lambda_{t+1}k_{t+1}^\theta h_{t+1}^{1-\theta} + (1-\delta)k_{t+1} - k_{t+2}} \mid \lambda_t\right]$$

and

$$(1-\theta)(1-h_t)\left(\lambda_t k_t^\theta h_t^{-\theta}\right) = A\left(\lambda_t k_t^\theta h_t^{1-\theta} + (1-\delta)k_t - k_{t+1}\right)$$

- In equilibrium,

$$c_t = \lambda_t k_t^\theta h_t^{1-\theta} + (1-\delta)k_t - k_{t+1}$$

Simplifying the first order conditions (continued)

Factor markets give

$$r_t = \theta\lambda_t k_t^{\theta-1} h_t^{1-\theta}$$

and

$$w_t = (1-\theta)\lambda_t k_t^\theta h_t^{-\theta}$$

First order conditions are simply

$$\frac{1}{c_t} = \beta E_t\left[\frac{r_{t+1} + (1-\delta)}{c_{t+1}} \mid \lambda_t\right]$$

and

$$(1-h_t)\,w_t = Ac_t$$

Stationary states

- Stationary state value of $\overline{h} = h_t = h_{t+1}$ is

$$\overline{h} = \frac{1}{1 + \frac{A}{(1-\theta)}\left[1 - \frac{\beta\delta\theta}{1-\beta(1-\delta)}\right]},$$

- Stationary state value of $\overline{k} = k_t = k_{t+1} = k_{t+2}$ is

$$\overline{k} = \overline{h}\left[\frac{\theta\overline{\lambda}}{\frac{1}{\beta} - (1-\delta)}\right]^{\frac{1}{1-\theta}}.$$

How to study dynamics

1. Find the approximate Value function and Plan
   (a) These will describe the dynamics within the precision of the approximation
   (b) Can be complicated to find
      i. Especially if the domain of stochastic variable is large
   (c) Can be impossible
      i. If the model is not single agent
      ii. If the model can not be approximated by social planner
2. Alternative approachs
   (a) Log linear approximation of the model
      i. After the optimization has been done
      ii. After equilibrium conditions have been imposed
   (b) Quadratic linear appoximation of the problem

Log-linearization techniques

- Consider a function of the form

$$F(x_t) = \frac{G(x_t)}{H(x_t)}$$

- Taking logs of both side gives

$$\ln(F(x_t)) = \ln(G(x_t)) - \ln(H(x_t))$$

- The first order Taylor series expansion
  – around the stationary state values $\overline{x}$

– gives

$$\ln(F(\overline{x})) + \frac{F'(\overline{x})}{F(\overline{x})}(x_t - \overline{x}) \approx \ln(G(\overline{x})) + \frac{G'(\overline{x})}{G(\overline{x})}(x_t - \overline{x}) - \ln(H(\overline{x})) - \frac{H'(\overline{x})}{H(\overline{x})}(x_t - \overline{x})$$

Log-linearization techniques (direct method)

- In the stationary state

$$\ln(F(\overline{x})) = \ln(G(\overline{x})) - \ln(H(\overline{x})))$$

- So the first order Taylor expansion can be written as

$$\frac{F'(\overline{x})}{F(\overline{x})}(x_t - \overline{x}) \approx \frac{G'(\overline{x})}{G(\overline{x})}(x_t - \overline{x}) - \frac{H'(\overline{x})}{H(\overline{x})}(x_t - \overline{x})$$

- Remember that this holds only near $\overline{x}$

An example using a Cobb-Douglas production function

$$Y_t = \lambda_t K_t^{\theta} H_t^{1-\theta}$$

- Take logs

$$\ln Y_t = \ln \lambda_t + \theta \ln K_t + (1-\theta) \ln H_t$$

- first order Taylor expansion gives

$$\ln \overline{Y} + \frac{1}{\overline{Y}}\left(Y_t - \overline{Y}\right) \approx \ln \overline{\lambda} + \frac{1}{\overline{\lambda}}\left(\lambda_t - \overline{\lambda}\right) + \theta \ln \overline{K} + \frac{\theta}{\overline{K}}\left(K_t - \overline{K}\right) + (1-\theta)\ln \overline{H} + \frac{(1-\theta)}{\overline{H}}\left(H_t - \overline{H}\right)$$

.

- Since in a stationary state

$$\ln \overline{Y} = \ln \overline{\lambda} + \theta \ln \overline{K} + (1-\theta) \ln \overline{H}$$

- get

$$\frac{1}{\overline{Y}}\left(Y_t - \overline{Y}\right) \approx \frac{1}{\overline{\lambda}}\left(\lambda_t - \overline{\lambda}\right) + \frac{\theta}{\overline{K}}\left(K_t - \overline{K}\right) + \frac{(1-\theta)}{\overline{H}}\left(H_t - \overline{H}\right)$$

- That reduces to

$$\frac{Y_t}{\overline{Y}} + 1 \approx \frac{\lambda_t}{\overline{\lambda}} + \frac{\theta K_t}{\overline{K}} + \frac{(1-\theta) H_t}{\overline{H}}$$

Log-linearization techniques (Uhlig's method)

- Write the original variable as

$$X_t = \overline{X} e^{\widetilde{X}_t}$$

or

$$\widetilde{X}_t = \ln X_t - \ln \overline{X}$$

- bring together all the exponential terms that you can

$$\frac{A_t B_t^\alpha}{C_t^\delta} = \frac{\overline{A} e^{\widetilde{A}_t} \overline{B}^\alpha e^{\alpha \widetilde{B}_t}}{\overline{C}^\delta e^{\delta \widetilde{C}_t}}$$

becomes

$$\frac{\overline{AB}^\alpha}{\overline{C}^\delta} e^{\widetilde{A}_t + \alpha \widetilde{B}_t - \delta \widetilde{C}_t}$$

Reference:Uhlig, Harald, (1999) "A toolkit for analysing nonlinear dynamic stochastic models easily", in Ramon Marimon and Andrew Scott, Eds., Computational Methods for the Study of Dynamic Economies, Oxford University Press, Oxford, p.30-61.

Log-linearization techniques (Uhlig's method)

- The Taylor series expansion (linear) gives

$$\begin{aligned} e^{\widetilde{A}_t + \alpha \widetilde{B}_t - \delta \widetilde{C}_t} &\approx e^{\widetilde{A} + \alpha \widetilde{B} - \delta \widetilde{C}} + e^{\widetilde{A} + \alpha \widetilde{B} - \delta \widetilde{C}} \left(\widetilde{A}_t - \widetilde{A} \right) \\ &\quad + \alpha e^{\widetilde{A} + \alpha \widetilde{B} - \delta \widetilde{C}} \left(\widetilde{B}_t - \widetilde{B} \right) - \delta e^{\widetilde{A} + \alpha \widetilde{B} - \delta \widetilde{C}} \left(\widetilde{C}_t - \widetilde{C} \right) \\ &= 1 + \widetilde{A}_t + \alpha \widetilde{B}_t - \delta \widetilde{C}_t, \end{aligned}$$

- So

$$e^{\widetilde{A}_t + \alpha \widetilde{B}_t - \delta \widetilde{C}_t} \approx 1 + \widetilde{A}_t + \alpha \widetilde{B}_t - \delta \widetilde{C}_t$$

- The approximation is

$$\frac{A_t B_t^\alpha}{C_t^\delta} \approx \frac{\overline{AB}^\alpha}{\overline{C}^\delta} \left(1 + \widetilde{A}_t + \alpha \widetilde{B}_t - \delta \widetilde{C}_t \right)$$

Log-linearization techniques (Uhlig's method)

- Some rules from Uhlig

$$\begin{aligned} e^{\widetilde{X}_t + a \widetilde{Y}_t} &\approx 1 + \widetilde{X}_t + a \widetilde{Y}_t, \\ \widetilde{X}_t \widetilde{Y}_t &\approx 0, \\ E_t \left[a e^{\widetilde{X}_{t+1}} \right] &\approx a + a E_t \left[\widetilde{X}_{t+1} \right] \\ E_t \left[X_{t+1} \right] &= \overline{X} \left(1 + E_t \left[\widetilde{X}_{t+1} \right] \right) \end{aligned}$$

Log linear version of Hansen's model

- The five equations of the Hansen model are (adjusted)

$$\begin{aligned}
1 &= \beta E_t \left[\frac{C_t}{C_{t+1}} \left(r_{t+1} + (1-\delta) \right) \right] \\
AC_t &= (1-\theta)(1-H_t) \frac{Y_t}{H_t} \\
C_t &= Y_t + (1-\delta) K_t - K_{t+1} \\
Y_t &= \lambda_t K_t^{\theta} H_t^{1-\theta} \\
r_t &= \theta \frac{Y_t}{K_t}
\end{aligned}$$

- We will do the log-linearization equation by equation

Log linear version of Hansen's model

- First equation

$$1 = \beta E_t \left[\frac{C_t}{C_{t+1}} \left(r_{t+1} + (1-\delta) \right) \right]$$

$$\begin{aligned}
1 &= \beta E_t \left[\frac{\overline{C} e^{\widetilde{C}_t}}{\overline{C} e^{\widetilde{C}_{t+1}}} \overline{r} e^{\widetilde{r}_{t+1}} + (1-\delta) \frac{\overline{C} e^{\widetilde{C}_t}}{\overline{C} e^{\widetilde{C}_{t+1}}} \right] \\
&= \beta E_t \left[\overline{r} e^{\widetilde{C}_t - \widetilde{C}_{t+1} + \widetilde{r}_{t+1}} + (1-\delta) e^{\widetilde{C}_t - \widetilde{C}_{t+1}} \right] \\
&\approx \beta \left(\overline{r} E_t \left[1 + \widetilde{C}_t - \widetilde{C}_{t+1} + \widetilde{r}_{t+1} \right] + (1-\delta) \left[1 + \widetilde{C}_t - \widetilde{C}_{t+1} \right] \right) \\
&= E_t \left[1 + \widetilde{C}_t - \widetilde{C}_{t+1} + \beta \overline{r} \widetilde{r}_{t+1} \right],
\end{aligned}$$

or (after cancelling the 1's and cleaning up the expections)

$$0 \approx \widetilde{C}_t - E_t \widetilde{C}_{t+1} + \beta \overline{r} E_t \widetilde{r}_{t+1}$$

Log linear version of Hansen's model

- Second equation

$$AC_t = (1-\theta)(1-H_t) \frac{Y_t}{H_t}$$

$$A\overline{C} e^{\widetilde{C}_t} = (1-\theta) \frac{\overline{Y}}{\overline{H}} e^{\widetilde{Y}_t - \widetilde{H}_t} - (1-\theta) \overline{Y} e^{\widetilde{Y}_t}$$

$$A\overline{C} \left(1 + \widetilde{C}_t \right) \approx (1-\theta) \frac{\overline{Y}}{\overline{H}} \left(1 + \widetilde{Y}_t - \widetilde{H}_t \right) - (1-\theta) \overline{Y} \left(1 + \widetilde{Y}_t \right)$$

$$A\overline{C}\widetilde{C}_t \approx \left[(1-\theta) \frac{\left(1-\overline{H}\right)\overline{Y}}{\overline{H}} \right] \widetilde{Y}_t - (1-\theta) \frac{\overline{Y}}{\overline{H}} \widetilde{H}_t$$

- given that in the stationary state

$$A\overline{C} = (1-\theta)\frac{\left(1-\overline{H}\right)\overline{Y}}{\overline{H}}$$

Log linear version of Hansen's model

- This becomes

$$\widetilde{C}_t \approx \widetilde{Y}_t - \frac{(1-\theta)\frac{\overline{Y}}{\overline{H}}}{(1-\theta)\frac{(1-\overline{H})\overline{Y}}{\overline{H}}}\widetilde{H}_t = \widetilde{Y}_t - \frac{\widetilde{H}_t}{1-\overline{H}}$$

- so

$$0 = \widetilde{C}_t - \widetilde{Y}_t + \frac{\widetilde{H}_t}{1-\overline{H}}$$

Log linear version of Hansen's model

- The next three equations (in their Log-linear form) are

$$0 \approx \overline{Y}\widetilde{Y}_t - \overline{C}\widetilde{C}_t + \overline{K}\left[(1-\delta)\widetilde{K}_t - \widetilde{K}_{t+1}\right]$$

$$0 \approx \widetilde{\lambda}_t + \theta\widetilde{K}_t + (1-\theta)\widetilde{H}_t - \widetilde{Y}_t$$

$$0 \approx \widetilde{Y}_t - \widetilde{K}_t - \widetilde{r}_t$$

- where $\overline{r} = \theta\overline{Y}/\overline{K}$

Log linear version of Hansen's model

- The stochastic process is

$$\lambda_{t+1} = \gamma\lambda_t + \varepsilon_{t+1}$$

- putting in the log difference of the λ's

$$\overline{\lambda}e^{\widetilde{\lambda}_{t+1}} = \gamma\overline{\lambda}e^{\widetilde{\lambda}_t} + \varepsilon_{t+1}$$

- the linerar approximation is

$$\overline{\lambda}\left(1+\widetilde{\lambda}_{t+1}\right) = \gamma\overline{\lambda}\left(1+\widetilde{\lambda}_t\right) + \varepsilon_{t+1}$$

- So the simple version is

$$\widetilde{\lambda}_{t+1} = \gamma\widetilde{\lambda}_t + \mu_{t+1}$$

The log-linear version of the model

- The equations of the full log-linear model are

$$\begin{aligned}
0 &= \widetilde{C}_t - E_t\widetilde{C}_{t+1} + \beta\overline{r}E_t\widetilde{r}_{t+1} \\
0 &= \widetilde{C}_t - \widetilde{Y}_t + \frac{\widetilde{H}_t}{1-\overline{H}} \\
0 &= \overline{Y}\widetilde{Y}_t - \overline{C}\widetilde{C}_t + \overline{K}\left[(1-\delta)\widetilde{K}_t - \widetilde{K}_{t+1}\right] \\
0 &= \widetilde{\lambda}_t + \theta\widetilde{K}_t + (1-\theta)\,\widetilde{H}_t - \widetilde{Y}_t \\
0 &= \widetilde{Y}_t - \widetilde{K}_t - \widetilde{r}_t
\end{aligned}$$

  and

$$\widetilde{\lambda}_{t+1} = \gamma\widetilde{\lambda}_t + \mu_{t+1}$$

Solving the log-linear version of the model

- The variables of the model are $\left\{ \widetilde{K}_{t+1} \quad \widetilde{Y}_t \quad \widetilde{C}_t \quad \widetilde{H}_t \quad \widetilde{r}_t \right\}$ plus the stochastic variables λ_t
- Define the state variables as

$$x_t = \left[\widetilde{K}_t\right]$$

- Define the "jump" variables as

$$y_t = \begin{bmatrix} Y_t \\ C_t \\ H_t \\ r_t \end{bmatrix}$$

- Define the stochastic variable as

$$z_t = [\lambda_t]$$

Solving the log-linear version of the model

- The model can be written as

$$\begin{aligned}
0 &= Ax_t + Bx_{t-1} + Cy_t + Dz_t, \\
0 &= E_t\left[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t\right], \\
z_{t+1} &= Nz_t + \varepsilon_{t+1}, \quad E_t(\varepsilon_{t+1}) = 0.
\end{aligned}$$

Where

$$A = \begin{bmatrix} 0 \\ -K \\ 0 \\ 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ K(-\delta+1) \\ \theta \\ -1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & -1 & -\frac{1}{1-\overline{H}} & 0 \\ \overline{Y} & -\overline{C} & 0 & 0 \\ -1 & 0 & 1-\theta & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

Solving the linear version of the model

$$D = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$F = [0], \quad G = [0], \quad H = [0],$$
$$J = \begin{bmatrix} 0 & -1 & 0 & \beta\overline{r} \end{bmatrix},$$
$$K = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\begin{aligned} L &= [0] \\ M &= [0] \\ N &= [\gamma] \end{aligned}$$

Solving the linear version of the model

- We look for a solution of the form

$$\begin{aligned} x_t &= Px_{t-1} + Qz_t \\ y_t &= Rx_{t-1} + Sz_t \end{aligned}$$

- Note that here C is of full rank and has a well defined inverse C^{-1}
- The solutions can be found from

$$\begin{aligned} 0 &= (F - JC^{-1}A)P^2 - (JC^{-1}B - G + KC^{-1}A)P - KC^{-1}B + H \\ R &= -C^{-1}(AP + B), \end{aligned}$$

$$\begin{aligned} &\left(N' \otimes (F - JC^{-1}A) + I_k \otimes (JR + FP + G - KC^{-1}A)\right) vec(Q) \\ = \; & vec\left((JC^{-1}D - L)N + KC^{-1}D - M\right) \end{aligned}$$

and

$$S = -C^{-1}(AQ + D)$$

Explaining the solution

- We look for the laws of motion of the model

$$\begin{aligned} x_t &= Px_{t-1} + Qz_t, \\ y_t &= Rx_{t-1} + Sz_t. \end{aligned}$$

- We begin by substituting the laws of motion into the two equations of the model
- Reduce each equation to one in which there are only two variables:

$$x_{t-1} \text{ and } z_t.$$

- Use the stochastic process in the expectational equation to replace

$$z_{t+1} = Nz_t + \varepsilon_{t+1}$$

- Taking expectations, the $\varepsilon_{t+1} = 0$ disappear

Explaining the solution

- Begin with the model

$$\begin{aligned} 0 &= Ax_t + Bx_{t-1} + Cy_t + Dz_t \\ 0 &= E_t \left[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t \right] \end{aligned}$$

- Substitute in

$$\begin{aligned} x_t &= Px_{t-1} + Qz_t, \\ y_t &= Rx_{t-1} + Sz_t. \end{aligned}$$

- In the first equation this gives

$$0 = A \left[Px_{t-1} + Qz_t \right] + Bx_{t-1} + C \left[Rx_{t-1} + Sz_t \right] + Dz_t$$

Explaining the solution

- In the second equation

$$\begin{aligned} 0 = {} & E_t \left[F \left[Px_t + Qz_{t+1} \right] + G \left[Px_{t-1} + Qz_t \right] + Hx_{t-1} \right. \\ & \left. + J \left[Rx_t + Sz_{t+1} \right] + K \left[Rx_{t-1} + Sz_t \right] + Lz_{t+1} + Mz_t \right] \end{aligned}$$

- Substitute one more time in the second equation

$$\begin{aligned} 0 = {} & E_t \left[F \left[P \left[Px_{t-1} + Qz_t \right] + Q \left[Nz_t + \varepsilon_{t+1} \right] \right] + G \left[Px_{t-1} + Qz_t \right] \right. \\ & + Hx_{t-1} + J \left[R \left[Px_{t-1} + Qz_t \right] + S \left[Nz_t + \varepsilon_{t+1} \right] \right] \\ & \left. + K \left[Rx_{t-1} + Sz_t \right] + L \left[Nz_t + \varepsilon_{t+1} \right] + Mz_t \right] \end{aligned}$$

- This simplifies to (because $E_t \varepsilon_{t+1} = 0$) and we remove the expectations operator

$$\begin{aligned} 0 = {} & F \left[P \left[Px_{t-1} + Qz_t \right] + QNz_t \right] + G \left[Px_{t-1} + Qz_t \right] \\ & + Hx_{t-1} + J \left[R \left[Px_{t-1} + Qz_t \right] + SNz_t \right] \\ & + K \left[Rx_{t-1} + Sz_t \right] + LNz_t + Mz_t \end{aligned}$$

Explaining the solution

- The two equations can be rearranged to give

$$0 = [AP + B + CR]\, x_{t-1} + [AQ + CS + D]\, z_t,$$

and

$$\begin{aligned} 0 &= [FPP + GP + H + JRP + KR]\, x_{t-1} \\ &\quad + [FPQ + FQN + GQ + JRQ + JSN + KS + LN + M]\, z_t. \end{aligned}$$

- Since these equations need to hold for all x_{t-1} and z_t, it must be that

$$\begin{aligned} 0 &= AP + B + CR \\ 0 &= AQ + CS + D \\ 0 &= FPP + GP + H + JRP + KR \\ 0 &= FPQ + FQN + GQ + JRQ + JSN + KS + LN + M \end{aligned}$$

Explaining the solution

- The third equation is

$$0 = FP^2 + GP + JRP + H + KR$$

and the first is (if the inverse of C exists)

$$R = -C^{-1}AP - C^{-1}B$$

- Combining these one gets

$$\begin{aligned} 0 &= FP^2 + GP - J\left[C^{-1}AP + C^{-1}B\right] P \\ &\quad + H - K\left[C^{-1}AP + C^{-1}B\right] \\ 0 &= FP^2 - JC^{-1}AP^2 + GP - JC^{-1}AP^2 - JC^{-1}BP \\ &\quad + H - KC^{-1}AP - KC^{-1}B \\ 0 &= \left[F - JC^{-1}A\right] P^2 - \left[JC^{-1}B + KC^{-1}A - G\right] P \\ &\quad - KC^{-1}B + H \end{aligned}$$

Explaining the solution

- Here F is a 1×1 matrix (a scalar)
- Finding the solution to the quadratic equation

$$\begin{aligned} 0 &= \left[F - JC^{-1}A\right] P^2 - \left[JC^{-1}B + KC^{-1}A - G\right] P \\ &\quad - KC^{-1}B + H \end{aligned}$$

can be done using

$$0 = aP^2 + bP + c$$

- The solution to this equation is found from

$$P = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- There are usually two different solutions to this problem. We use $|P| < 1$ in order to choose the stable root.
- Once P is known, finding R is simple using

$$R = -C^{-1}AP - C^{-1}B$$

Explaining the solution

- Finding Q (with P and R already known, from above)
- Use the equations

$$0 = FPQ + FQN + GQ + JRQ + JSN + KS + LN + M$$

  and

$$0 = AQ + CS + D$$

- S can be written as

$$S = -C^{-1}AQ - C^{-1}D$$

- Substitute this into the first equation

$$\begin{aligned} 0 &= FPQ + FQN + GQ + JRQ - JC^{-1}AQN - JC^{-1}DN \\ &\quad -KC^{-1}AQ - KC^{-1}D + LN + M \end{aligned}$$

- Rearrange to get

$$\begin{aligned} &\left[FP + G + JR - KC^{-1}A\right]Q + \left[F - JC^{-1}A\right]QN \\ = \; & JC^{-1}DN + KC^{-1}D - LN + M \end{aligned}$$

Explaining the solution

- This equation

$$\begin{aligned} &\left[FP + G + JR - KC^{-1}A\right]Q + \left[F - JC^{-1}A\right]QN \\ = \; & JC^{-1}DN + KC^{-1}D - LN + M \end{aligned}$$

  has Q in two different places on the left hand side
  - Q in the final position in $\left[FP + G + JR - KC^{-1}A\right]Q$
  - Q in the second to the last position in $\left[F - JC^{-1}A\right]QN$

- Need to use a theorem from advanced matrix algebra

Theorem 1 *Let* $\mathbf{A}$, $\mathbf{B}$, *and* $\mathbf{C}$ *be matrices whose dimensions are such that the product* $\mathbf{ABC}$ *exists. Then*

$$vec(\mathbf{ABC}) = (\mathbf{C}' \otimes \mathbf{A}) \cdot vec(\mathbf{B})$$

where the symbol $\otimes$ *denotes the Kronecker product.*

Explaining the solution

- Think of

$$\begin{aligned} & \left[FP + G + JR - KC^{-1}A\right] Q + \left[F - JC^{-1}A\right] QN \\ = \; & JC^{-1}DN + KC^{-1}D - LN + M \end{aligned}$$

as

$$WQI + XQN = Z$$

(notice that we added I) where

$$\begin{aligned} W &= FP + G + JR - KC^{-1}A \\ X &= F - JC^{-1}A \\ Z &= JC^{-1}DN + KC^{-1}D - LN + M \end{aligned}$$

.

- Take *vec* of both sides of the equation, so

$$vec\,(WQI) + vec\,(XQN) = vec\,(Z)$$

- This equals

$$(I' \otimes W)\, vec\,(Q) + (N' \otimes X)\, vec\,(Q) = vec\,(Z)$$

or

$$(I' \otimes W + N' \otimes X)\, vec\,(Q) = vec\,(Z)$$

- If $(I' \otimes W + N' \otimes X)$ is invertible

$$vec\,(Q) = (I' \otimes W + N' \otimes X)^{-1}\, vec\,(Z)$$

Explaining the solution

- What are *vec* and $\otimes$ (the Kronecker product)

- First *vec*

$$vec\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\right) = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{12} \\ a_{22} \\ a_{13} \\ a_{23} \end{bmatrix}.$$

- the columns are made into a vector

Explaining the solution

- The Kronecker product is

$$\begin{aligned} \mathbf{A} \otimes \mathbf{B} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{11}b_{31} & a_{11}b_{32} & a_{12}b_{31} & a_{12}b_{32} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \\ a_{21}b_{31} & a_{21}b_{32} & a_{22}b_{31} & a_{22}b_{32} \end{bmatrix}. \end{aligned}$$

Calibration

- Solution to model is numerical
- Need values for parameters
- Some we borrow from literature (quarterly)
  - $\beta = .99$
  - $\delta = .025$
  - $\theta = .36$
- Need a value for A
  - Choose A so that $\overline{H} = 1/3$
  - Use stationary state equation for $\overline{H}$

$$\overline{H} = \frac{1}{1 + \frac{A}{(1-\theta)}\left[1 - \frac{\beta\delta\theta}{1-\beta(1-\delta)}\right]}$$

  - $A = 1.72$ for $\overline{H} = .3335$
- $\overline{K} = 12.6695$ and using the production function, $\overline{Y} = 1.2353$

- $\overline{r} = 1/\beta = 1.0101$
- From data for US use $\gamma = .95$

Matices for Calibrated model

$$A = \begin{bmatrix} 0 \\ -12.670 \\ 0 \\ 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 12.353 \\ 0.36 \\ -1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & -1 & -1.5004 & 0 \\ 1.2353 & -0.9186 & 0 & 0 \\ -1 & 0 & .64 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Matices for Calibrated model

$$\begin{aligned} F &= [0] \\ G &= [0] \\ H &= [0] \end{aligned}$$

$$J = \begin{bmatrix} 0 & -1 & 0 & .0348 \end{bmatrix}$$
$$K = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} L &= [0] \\ M &= [0] \\ N &= [.95] \end{aligned}$$

Numerical solution for model

- The quadratic equation gives the solutions

$$P = 1.0592 \text{ and } P = 0.9537$$

- The stable value is

$$P = 0.9537$$

- The value for Q is

$$Q = 0.1132$$

- The matrices R and S are

$$R = \begin{bmatrix} 0.204\,5 \\ 0.569\,1 \\ -0.243 \\ -0.795\,5 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 1.\,452\,3 \\ 0.392 \\ 0.706\,7 \\ 1.\,452\,3 \end{bmatrix}$$

Numerical solution for model

- The laws of motion are

$$\begin{aligned} \widetilde{K}_{t+1} &= 0.9537\widetilde{K}_t + 0.1132\widetilde{\lambda}_t, \\ \widetilde{Y}_t &= 0.2045\widetilde{K}_t + 1.4523\widetilde{\lambda}_t, \\ \widetilde{C}_t &= 0.5691\widetilde{K}_t + 0.3920\widetilde{\lambda}_t, \\ \widetilde{H}_t &= -0.2430\widetilde{K}_t + 0.7067\widetilde{\lambda}_t, \\ \widetilde{r}_t &= -0.7955\widetilde{K}_t + 1.4523\widetilde{\lambda}_t. \end{aligned}$$

- Recall that $\widetilde{\lambda}_t$ follows the process

$$\widetilde{\lambda}_t = .95\widetilde{\lambda}_{t-1} + \mu_t$$

Two ways of finding the variances of the variables of the model

- Simulations
  - Run lots of simulated economies
  - Calculate the variances from this "data"
- Calculate variances from laws of motion
  - See book for detains
- Need to calibrate $var(\mu_t)$ so that $var(\widetilde{Y}_t) = 1.76\%$
  - gets standard error of $\mu_t = .0032$

Tables of second moments

- Standard errors as fraction of output

	$\widetilde{Y}_t$	$\widetilde{C}_t$	$\widetilde{H}_t$	$\widetilde{r}_t$	$\widetilde{I}_t$
Standard error	$5.484\sigma_\varepsilon$	$4.065\sigma_\varepsilon$	$1.640\sigma_\varepsilon$	$3.492\sigma_\varepsilon$	$11.742\sigma_\varepsilon$
As % of output	100%	74.1 2%	29.9 0%	63.6 7%	214.1%

- Standard errors from the data

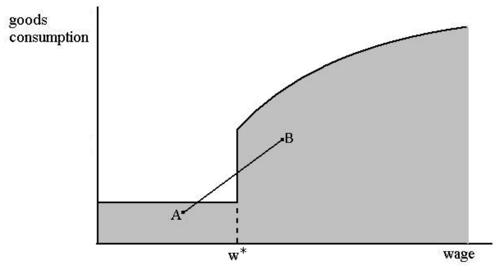
	$\widetilde{Y}_t$	$\widetilde{C}_t$	$\widetilde{H}_t$	$\widetilde{I}_t$
As % of output	100%	73.30%	94.3 2%	4 88.6 4%

- Does well for consumption
- Badly for hours worked and investment

```
% stationary state values are found in another program
A=[0 -kbar 0 0]';
B=[0 (1-delta)*kbar theta -1]';
C=[1 -1 -1/(1-hbar) 0
 ybar -cbar 0 0
 -1 0 1-theta 0
 1 0 0 -1];
D=[0 0 1 0]';
F=[0];
G=F;
H=F;
J=[0 -1 0 beta*rbar];
K=[0 1 0 0];
L=F;
M=F;
N=[.95];
Cinv=inv(C);
a=F-J*Cinv*A;
b=-(J*Cinv*B-G+K*Cinv*A);
c=-K*Cinv*B+H;
P1=(-b+sqrt(b^2-4*a*c))/(2*a);
P2=(-b-sqrt(b^2-4*a*c))/(2*a);
if abs(P1)<1
 P=P1;
else
 P=P2;
end
R=-Cinv*(A*P+B);
Q=(J*Cinv*D-L)*N+K*Cinv*D-M;
QD=kron(N',(F-J*Cinv*A))+(J*R+F*P+G-K*Cinv*A);
Q=Q/QD;
S=-Cinv*(A*Q+D);
```

Hansen's model with indivisible labor

- Objective: increase variance of hours worked
- Make labor indivisible
  - one works X hours per week or not at all
- Add unemployment
  - since some fraction of the population will not be working



Problem of non-convexity of consumption set

- In general, maximization is only valid over convex sets
- Def of a convex set
  - straight lines between any two points in set are also in set
- Example of a non-convex set

How non-convexity is fixed in Hansen's model

- The problem is the jump in income
  - between working and not working
- Hansen invented an "unemployment insurance"
- Lump sum transfers that make income equal for all
  - solves non-convexity problem
  - consumption increases smoothly with wage
  - since all receive same income (based on wages)
  - solve problem of too much heterogenity

Household problem

- maximize

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, \alpha_t)$$

subject to

$$c_t + i_t = w_t h_t + r_t k_t$$

α_t = probability in time t of supplying h_0 units of labor

- Expected utility

$$
\begin{aligned}
u(c_t, \alpha_t) &= \ln c_t + h_t \frac{A \ln(1-h_0)}{h_0} + A(1 - \frac{h_t}{h_0}) \ln(1) \\
u(c_t, \alpha_t) &= \ln c_t + h_t \frac{A \ln(1-h_0)}{h_0}
\end{aligned}
$$

Household problem

- Maximization problem becomes

$$\max \sum_{t=0}^{\infty} \beta^t \left[\ln c_t + B h_t\right]$$

with

$$B = \frac{A \ln(1-h_0)}{h_0}$$

- subject to constraints

$$\lambda_t k_t^{\theta} h_t^{1-\theta} = c_t + k_{t+1} - (1-\delta) k_t$$

and

$$\ln \lambda_{t+1} = \gamma \ln \lambda_t + \varepsilon_{t+1}$$

Household problem

- First order conditions

$$
\begin{aligned}
0 &= \frac{1}{c_t} \left((1-\theta) \lambda_t k_t^{\theta} h_t^{-\theta} \right) + B, \\
0 &= -\frac{1}{c_t} + E_t \left[\frac{1}{c_{t+1}} \theta \lambda_{t+1} k_t^{\theta-1} h_t^{1-\theta} + (1-\delta) \right]
\end{aligned}
$$

- With equilibrium condition, these simplify to

$$
\begin{aligned}
1 &= \beta E_t \left[\frac{C_t}{C_{t+1}} \left(r_{t+1} + (1-\delta) \right) \right], \\
C_t &= -\frac{(1-\theta) Y_t}{B H_t}.
\end{aligned}
$$

Full model

$$
\begin{aligned}
1 &= \beta E_t \left[\frac{C_t}{C_{t+1}} \left(r_{t+1} + (1-\delta) \right) \right] \\
C_t &= -\frac{(1-\theta) Y_t}{B H_t}
\end{aligned}
$$

$$
\begin{aligned}
C_t + K_{t+1} &= Y_t + (1-\delta)K_t \\
r_t &= \theta\lambda_t K_t^{\theta-1} H_t^{1-\theta} \\
Y_t &= \lambda_t K_t^{\theta} H_t^{1-\theta}
\end{aligned}
$$

Stationary state

- Equations

$$
\begin{aligned}
\frac{1}{\beta} &= \overline{r} + (1-\delta) \\
\overline{C} &= -\frac{(1-\theta)\,\overline{Y}}{B\overline{H}} \\
\overline{r} &= \theta\overline{K}^{\theta-1}\overline{H}^{1-\theta} \\
\overline{Y} &= \overline{K}^{\theta}\overline{H}^{1-\theta} \\
\overline{C} &= \overline{Y} - \delta\overline{K}
\end{aligned}
$$

- solve to give

$$
\overline{H} = -\frac{(1-\theta)}{B\left(1 - \frac{\delta\theta\beta}{1-\beta(1-\delta)}\right)} \quad \text{and} \quad \overline{K} = \left[\frac{\theta\beta}{1+\beta(1-\delta)}\right]^{\frac{1}{1-\theta}} \overline{H}
$$

Stationary state
Comparing to basic Hansen model

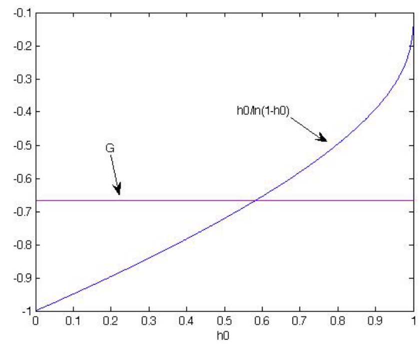
- To get same stationary state, need $\overline{H}$ the same in both cases
- Then other variables will be the same
- Old stationary state equation

$$
\overline{H} = \frac{1}{1 + \frac{A}{(1-\theta)}\left[1 - \frac{\beta\delta\theta}{1-\beta(1-\delta)}\right]}
$$

- Set the two equal

$$
\frac{1}{1 + \frac{A}{(1-\theta)}\left[1 - \frac{\beta\delta\theta}{1-\beta(1-\delta)}\right]} = -\frac{(1-\theta)}{\frac{A\ln(1-h_0)}{h_0}\left(1 - \frac{\delta\theta\beta}{1-\beta(1-\delta)}\right)},
$$

- We replaced B with $\frac{A\ln(1-h_0)}{h_0}$
- Need to determine h_0 that make the two SS the same



Stationary state

- Solve to get

$$\frac{h_0}{\ln(1-h_0)} = -\frac{\frac{A}{(1-\theta)}\left[1-\frac{\beta\delta\theta}{1-\beta(1-\delta)}\right]}{1+\frac{A}{(1-\theta)}\left[1-\frac{\beta\delta\theta}{1-\beta(1-\delta)}\right]} = G$$

- G is a constant
- To find h_0
- Get $h_0 = .583$, $\overline{\alpha} = .573$, and $\overline{H} = .3335$

Log-linear model

- Taking the log-linear approximation of the model gives

$$\begin{aligned}
0 &\approx \widetilde{C}_t - E_t\widetilde{C}_{t+1} + \beta\overline{r}E_t\widetilde{r}_{t+1} \\
0 &\approx \widetilde{C}_t + \widetilde{H}_t - \widetilde{Y}_t \\
0 &\approx \overline{Y}\widetilde{Y}_t - \overline{C}\widetilde{C}_t + (1-\delta)\overline{K}\widetilde{K}_t - \overline{K}\widetilde{K}_{t+1} \\
0 &\approx \widetilde{Y}_t - \widetilde{\lambda}_t - \theta\widetilde{K}_t - (1-\theta)\widetilde{H}_t \\
0 &\approx \widetilde{Y}_t - \widetilde{K}_t - \widetilde{r}_t
\end{aligned}$$

Solution method

- Use Uhlig's method

$$\begin{aligned}
0 &= Ax_t + Bx_{t-1} + Cy_t + Dz_t, \\
0 &= E_t\left[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t\right], \\
z_{t+1} &= Nz_t + \varepsilon_{t+1}, \quad E_t(\varepsilon_{t+1}) = 0,
\end{aligned}$$

where, $x_t = \left[\widetilde{K}_t\right]$, $y_t = \left[\widetilde{Y}_t, \widetilde{C}_t, \widetilde{H}_t, \widetilde{r}_t\right]'$, and $z_t = \left[\widetilde{\lambda}_t\right]$

- Solve for

$$
\begin{aligned}
x_t &= Px_{t-1} + Qz_t \\
y_t &= Rx_{t-1} + Sz_t
\end{aligned}
$$

.

$$A = \begin{bmatrix} 0 \\ -K \\ 0 \\ 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ K(-\delta + 1) \\ \theta \\ -1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & -1 & -1 & 0 \\ \overline{Y} & -\overline{C} & 0 & 0 \\ -1 & 0 & (1-\theta) & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$F = [0], \quad G = [0], \quad H = [0]$$

$$J = \begin{bmatrix} 0 & -1 & 0 & \beta\overline{r} \end{bmatrix} \qquad K = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

$$L = [0], \quad M = [0], \text{ and } N = [\gamma].$$

Results

- The linear policy functions are

$$\widetilde{K}_{t+1} = .9418\widetilde{K}_t + .1552\lambda_t$$

and

$$y_t = R\widetilde{K}_t + S\lambda_t$$

where

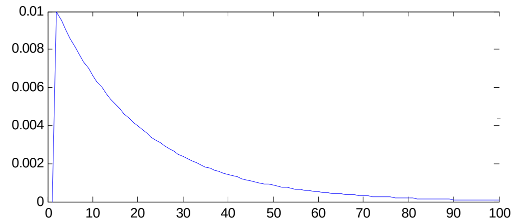
$$R = \begin{bmatrix} 0.055 \\ 0.531\,6 \\ -0.476\,6 \\ -0.945 \end{bmatrix} \qquad S = \begin{bmatrix} 1.941\,8 \\ 0.470\,3 \\ 1.471\,5 \\ 1.941\,7 \end{bmatrix}$$

Results

- Using this model, we calculate the variances of the variables
-

	$\widetilde{Y}_t$	$\widetilde{C}_t$	$\widetilde{H}_t$	$\widetilde{r}_t$	$\widetilde{I}_t$
Standard errors	$6.431\sigma_\varepsilon$	$4.081\sigma_\varepsilon$	$3.444\sigma_\varepsilon$	$4.514\sigma_\varepsilon$	$15.722\sigma_\varepsilon$
As % of output	100%	63.4 6%	53.5 5%	70.1 9%	244.5%

- Increased variance in hours worked
- Slight increase in investment
- Lower variance in consumption (compared to data)



Impulse response functions

- How does the economy respond to a one time shock to technology
- $\varepsilon_t = 0$ except $\varepsilon_2 = .01$ Recall that $\gamma = .95$

$$\widetilde{\lambda}_t = \gamma \widetilde{\lambda}_{t-1} + \varepsilon_t$$

- Response of technology (path of $\widetilde{\lambda}_t$)

Impulse response functions

- Then calculate the time path of capital with $\widetilde{K}_1 = 0$ using

$$\widetilde{K}_{t+1} = P\widetilde{K}_t + Q\widetilde{\lambda}_t$$

- get path of $\widetilde{K}_t$. Use this to find path of other variables using

$$y_t = R\widetilde{K}_t + S\lambda_t$$

Impulse response functions: Basic Hansen model
Impulse response functions: Hansen with indivisible labor
Comparing impulse responses

- Both models get same impulse
- Put each set of responses on different axis
- Get

Comparing impulse response

- Rotating so that we don't see the time axis

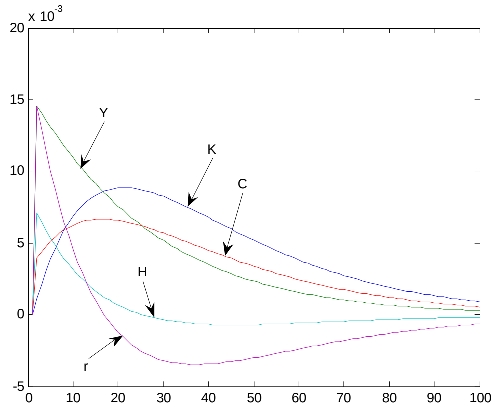


Figure 1: Responses of Hansen's basic model

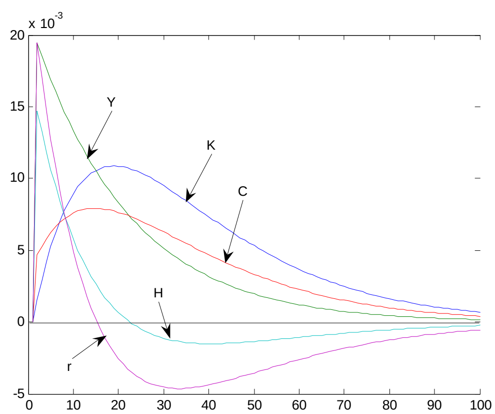


Figure 2: Responses for Hansen's model with indivisible labor

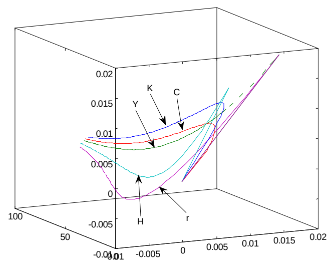


Figure 3: Responses for both Hansen models

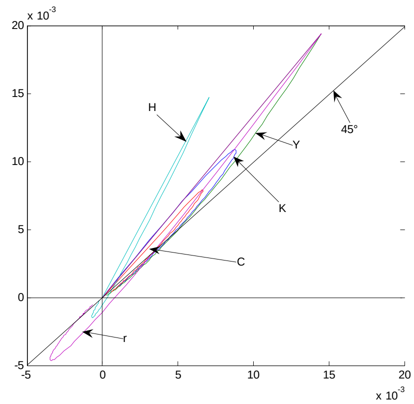


Figure 4: Comparing the response of the two models