1 Linear quadratic models

Linear-quadratic methods

- Alternative way to approximate models
- Results in linear approximation of a policy function
- Approximation is done when setting up the problem
- The objective of the Bellman equation is quadratic

The linear quadratic problem

- discounted quadratic objective function we are looking for is of the form

$$\sum_{t=0}^{\infty} \beta^t \left[x_t' R x_t + y_t' S y_t + 2 y_t' W x_t\right]$$

- subject to the linear budget constraints

$$x_{t+1} = A x_t + B y_t$$

- where
  - x_t is the $n \times 1$ vector of state variables,
  - y_t is a $m \times 1$ vector of control variables,
  - R and A are $n \times n$ matrices,
  - S is an $m \times m$ matrix,
  - and W and B are $m \times n$ matrices.

Second order Taylor approximations

Theorem 1 *Suppose that f is a function with domain D in $\mathbf{R}^p$ and range in $\mathbf{R}$, and suppose that f has continuous partial derivatives of order n in a neighborhood of every point on a line segment joining two points u, v in D. Then there exists a point $\widetilde{u}$ on this line segment such that*

$$\begin{aligned} f(v) &= f(u) + \frac{1}{1!} Df(u)(v-u) + \frac{1}{2!} D^2 f(u)(v-u)^2 \\ &\quad + \cdots + \frac{1}{(n-1)!} D^{n-1} f(u)(v-u)^{n-1} + \frac{1}{n!} D^n f(\widetilde{u})(v-u)^n. \end{aligned}$$

Second order Taylor approximations: example

- For the discounted utility function of the form

$$\sum_{t=0}^{\infty} \beta^t u(c_t, h_t) = \sum_{t=0}^{\infty} \beta^t \left[\ln c_t + A \ln(1-h_t)\right]$$

- the objective function is

$$\ln c_t + A \ln(1 - h_t)$$

- the first derivative of the objective function is the vector,

$$\begin{bmatrix} \frac{1}{c_t} & -\frac{A}{1-h_t} \end{bmatrix},$$

- the second derivative is the matrix,

$$\begin{bmatrix} -\frac{1}{c_t^2} & 0 \\ 0 & \frac{A}{(1-h_t)^2} \end{bmatrix}.$$

Second order Taylor approximations: example

- Taylor expansion (round $\bar{c}$ and $\bar{h}$) is

$$\begin{aligned} u(c_t, h_t) \quad &\approx \quad \ln \bar{c} + A \ln(1 - \bar{h}) + \begin{bmatrix} \frac{1}{\bar{c}} & -\frac{A}{1-\bar{h}} \end{bmatrix} \begin{bmatrix} c_t - \bar{c} \\ h_t - \bar{h} \end{bmatrix} \\ &+ \frac{1}{2} \begin{bmatrix} c_t - \bar{c} & h_t - \bar{h} \end{bmatrix} \begin{bmatrix} -\frac{1}{\bar{c}^2} & 0 \\ 0 & \frac{A}{\left(1-\bar{h}\right)^2} \end{bmatrix} \begin{bmatrix} c_t - \bar{c} \\ h_t - \bar{h} \end{bmatrix} \end{aligned}$$

- How to arrange this result so that it looks like

$$x_t' R x_t + y_t' S y_t + 2 y_t' W x_t$$

Method of Kydland and Prescott (General version)

- A general version is to maximize

$$\sum_{t=0}^{\infty} \beta^t F(x_t, y_t)$$

- subject to the linear budget constraint

$$x_{t+1} = G(x_t, y_t) = A x_t + B y_t$$

- where x_t are the period t state variables and y_t are the period t control variables.

- The second order Taylor expansion of the function $F(x_t, y_t)$ is

$$\begin{aligned} F(x_t, y_t) \quad &\approx \quad F(\bar{x}, \bar{y}) + \begin{bmatrix} F_x(\bar{x}, \bar{y}) & F_y(\bar{x}, \bar{y}) \end{bmatrix} \begin{bmatrix} x_t - \bar{x} \\ y_t - \bar{y} \end{bmatrix} \\ &+ \begin{bmatrix} x_t - \bar{x} & y_t - \bar{y} \end{bmatrix} \begin{bmatrix} \frac{F_{xx}(\bar{x},\bar{y})}{2} & \frac{F_{xy}(\bar{x},\bar{y})}{2} \\ \frac{F_{yx}(\bar{x},\bar{y})}{2} & \frac{F_{yy}(\bar{x},\bar{y})}{2} \end{bmatrix} \begin{bmatrix} x_t - \bar{x} \\ y_t - \bar{y} \end{bmatrix}. \end{aligned}$$

Method of Kydland and Prescott (General version)

- define a vector z_t

$$z_t = \begin{bmatrix} 1 \\ x_t \\ y_t \end{bmatrix}$$

- its value in the stationary state

$$\overline{z} = \begin{bmatrix} 1 \\ \overline{x} \\ \overline{y} \end{bmatrix}$$

- vector x_t is of length k
- the vector y_t of length l,
- the vector z_t is of length $1 + k + l$

Method of Kydland and Prescott (General version)

- Consider the $(1 + k + l) \times (1 + k + l)$ matrix

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

- The matrix m_{11} is 1×1, m_{22} is $k \times k$, m_{33} is $l \times l$, and the rest of the matrices conform to make M square.
- The product

$$\begin{aligned} z_t' M z_t &= m_{11} + (m_{12} + m_{21}')x_t + (m_{13} + m_{31}')y_t \\ &\quad + x_t' m_{22} x_t + x_t'(m_{23} + m_{32}')y_t + y_t' m_{33} y_t. \end{aligned}$$

Method of Kydland and Prescott (General version)

- Put all the constant components of the Taylor expansion into

$$\begin{aligned} m_{11} &= F(\overline{x}, \overline{y}) - F_x(\overline{x}, \overline{y})\overline{x} - F_y(\overline{x}, \overline{y})\overline{y} \\ &\quad + \frac{F_{xx}(\overline{x}, \overline{y})\overline{x}^2}{2} + F_{xy}(\overline{x}, \overline{y})\overline{xy} + \frac{F_{yy}(\overline{x}, \overline{y})\overline{y}^2}{2} \end{aligned}$$

- Define

$$m_{12} = m_{21}' = \frac{F_x(\overline{x}, \overline{y}) - \overline{x} F_{xx}(\overline{x}, \overline{y}) - \overline{y} F_{xy}(\overline{x}, \overline{y})}{2}$$

  and

$$m_{13} = m_{31}' = \frac{F_y(\overline{x}, \overline{y}) - \overline{x} F_{xy}(\overline{x}, \overline{y}) - \overline{y} F_{yy}(\overline{x}, \overline{y})}{2}$$

- These last two equations include all the linear components of the Taylor expansion in M and M a symmetric matrix

Method of Kydland and Prescott (General version)

- The quadratic components of the Taylor expansion are found in

$$m_{22} = \frac{F_{xx}(\overline{x}, \overline{y})}{2}$$

$$m_{23} = m_{32}' = \frac{F_{xy}(\overline{x}, \overline{y})}{2}$$

and

$$m_{33} = \frac{F_{yy}(\overline{x}, \overline{y})}{2}$$

- The quadratic discounted dynamic programming problem to be solved is

$$\sum_{t=0}^{\infty} \beta^t z_t' M z_t$$

with $z_t' = \begin{bmatrix} 1 & x_t & y_t \end{bmatrix}$, subject to the budget constraints

$$x_{t+1} = Ax_t + By_t$$

Method of Kydland and Prescott (Hansens model)

- A specific example of the problem to be solved is

$$\sum_{t=0}^{\infty} \beta^t u(c_t, h_t)$$

subject to the budget constraint

$$c_t = f(k_t, h_t) + (1 - \delta)k_t - k_{t+1}.$$

- This budget constraint is not linear, rewrite problem as

$$\max \sum_{t=0}^{\infty} \beta^t u(f(k_t, h_t) + (1 - \delta)k_t - k_{t+1}, h_t),$$

subject to the linear budget constraint

$$k_{t+1} = k_{t+1}$$

- The controls are k_{t+1} and h_t

Method of Kydland and Prescott (Hansens model)

- The exact problem is

$$\max \sum_{t=0}^{\infty} \beta^t \left[\ln \left(k_t^{\theta} h_t^{1-\theta} + (1-\delta) k_t - k_{t+1} \right) + A \ln (1 - h_t) \right],$$

subject to the linear budget constraint: $k_{t+1} = k_{t+1}$

- The quadratic Taylor expansion of the objective function is

$$\begin{aligned} u(\cdot) \approx\ & \ln \left(f(\overline{k}, \overline{h}) - \delta \overline{k} \right) + A \ln (1 - \overline{h}) \\ & + \frac{1}{\overline{c}} \left[\theta \frac{\overline{y}}{\overline{k}} + (1-\delta) \right] \left(k_t - \overline{k} \right) - \frac{1}{\overline{c}} \left(k_{t+1} - \overline{k} \right) \\ & + \left[(1-\theta) \frac{1}{\overline{c}} \frac{\overline{y}}{\overline{h}} - \frac{A}{1 - \overline{h}} \right] \left(h_t - \overline{h} \right) \\ & + \begin{bmatrix} \left(k_t - \overline{k} \right) \\ \left(k_{t+1} - \overline{k} \right) \\ \left(h_t - \overline{h} \right) \end{bmatrix}' \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \left(k_t - \overline{k} \right) \\ \left(k_{t+1} - \overline{k} \right) \\ \left(h_t - \overline{h} \right) \end{bmatrix} \end{aligned}$$

Method of Kydland and Prescott (Hansens model)

- where

$$a_{11} = -\frac{1}{2\overline{c}^2} \left[\theta \frac{\overline{y}}{\overline{k}} + (1-\delta) \right]^2 - \frac{1}{2\overline{c}} \theta (1-\theta) \frac{\overline{y}}{\overline{k}^2}$$

$$a_{12} = a_{21} = \frac{1}{2\overline{c}^2} \left[\theta \frac{\overline{y}}{\overline{k}} + (1-\delta) \right]$$

$$a_{13} = a_{31} = -\frac{1}{2\overline{c}^2} \left[\theta \frac{\overline{y}}{\overline{k}} + (1-\delta) \right] (1-\theta) \frac{\overline{y}}{\overline{h}} + \frac{1}{2\overline{c}} \theta (1-\theta) \frac{\overline{y}}{\overline{kh}}$$

$$a_{22} = -\frac{1}{2\overline{c}^2}$$

$$a_{23} = a_{32} = \frac{1}{2\overline{c}^2} (1-\theta) \frac{\overline{y}}{\overline{h}}$$

and

$$a_{33} = -\frac{1}{2\overline{c}^2} \left[(1-\theta) \frac{\overline{y}}{\overline{h}} \right]^2 - \frac{1}{2\overline{c}} \theta (1-\theta) \frac{\overline{y}}{\overline{h}^2} - \frac{A}{2 \left(1 - \overline{h} \right)^2}$$

Method of Kydland and Prescott (Hansens model)

- Define the four element vector $z_t = \begin{bmatrix} 1 & k_t & k_{t+1} & h_t \end{bmatrix}'$.
- The 4×4 matrix M is

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & a_{11} & a_{12} & a_{13} \\ m_{31} & a_{21} & a_{22} & a_{23} \\ m_{41} & a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- m_{11} contains all the constants,

$$
\begin{aligned}
m_{11} \quad = \quad & \ln\left(f(\overline{k}, \overline{h}) - \delta \overline{k}\right) + A \ln(1 - \overline{h}) \\
& -\frac{1}{\overline{c}} \left[\theta \frac{\overline{y}}{\overline{k}} + (1-\delta) - 1\right] \overline{k} - \left[(1-\theta)\,\frac{1}{\overline{c}} \frac{\overline{y}}{\overline{h}} - \frac{A}{1-\overline{h}}\right] \overline{h} \\
& + \begin{bmatrix} \overline{k} \\ \overline{k} \\ \overline{h} \end{bmatrix}' \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \overline{k} \\ \overline{k} \\ \overline{h} \end{bmatrix},
\end{aligned}
$$

Method of Kydland and Prescott (Hansens model)

- All the linear parts are in

$$
m_{12} = m_{21} = \frac{1}{\overline{c}} \left[\theta \frac{\overline{y}}{\overline{k}} + (1-\delta)\right] - \begin{bmatrix} \overline{k} & \overline{k} & \overline{h} \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}
$$

$$
m_{13} = m_{31} = -\frac{1}{\overline{c}} - \begin{bmatrix} \overline{k} & \overline{k} & \overline{h} \end{bmatrix} \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}
$$

and

$$
m_{14} = m_{41} = \left[(1-\theta)\,\frac{1}{\overline{c}} \frac{\overline{y}}{\overline{h}} - \frac{A}{1-\overline{h}}\right] - \begin{bmatrix} \overline{k} & \overline{k} & \overline{h} \end{bmatrix} \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}
$$

Method of Kydland and Prescott (Hansens model)

- The model can now be written as

$$
\sum_{t=0}^{\infty} \beta^t z_t' M z_t
$$

- subject to the budget constraint

$$
\begin{bmatrix} 1 \\ k_{t+1} \end{bmatrix} = A \begin{bmatrix} 1 \\ k_t \end{bmatrix} + B \begin{bmatrix} k_{t+1} \\ h_t \end{bmatrix}
$$

- where here, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

Solving the Quadratic Bellman equation

- Use $z_t \equiv \begin{bmatrix} x_t \\ y_t \end{bmatrix}$. let the first element of x_t be the constant 1.

- one wants to maximize
$$\sum_{t=0}^{\infty} \beta^t z_t' M z_t$$
subject to the linear budget constraint,
$$x_{t+1} = Ax_t + By_t$$

- The objective function is of the form
$$z_t' M z_t = \begin{bmatrix} x_t' & y_t' \end{bmatrix} \begin{bmatrix} R & W' \\ W & Q \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix},$$
where x_t is a $1 \times n$ vector, y_t is a $1 \times m$ vector, z_t is therefore a $1 \times (n+m)$ vector. The matrix R is $n \times n$, Q is $m \times m$, and W is $m \times n$.

Solving the Quadratic Bellman equation

- Since $x_t' W' y_t = y_t' W x_t$, this objective function can be written as
$$x_t' R x_t + y_t' Q y_t + 2 y_t' W x_t$$

- Based on this objective function, we look for a value function matrix P such that
$$x_t' P x_t = \max_{y_t} \left[z_t' M z_t + \beta x_{t+1}' P x_{t+1} \right]$$
subject to the linear budget constraints
$$x_{t+1} = Ax_t + By_t$$

- This Bellman equation can be written as
$$x_t' P x_t = \max_{y_t} \left[x_t' R x_t + y_t' Q y_t + 2 y_t' W x_t + \beta \left(Ax_t + By_t \right)' P \left(Ax_t + By_t \right) \right]$$

Solving the Quadratic Bellman equation

- The first order conditions from the maximization problem are
$$\left[Q + \beta B' P B \right] y_t = - \left[W + \beta B' P A \right] x_t,$$
which gives the policy function (matrix), F,
$$y_t = F x_t = - \left[Q + \beta B' P B \right]^{-1} \left[W + \beta B' P A \right] x_t.$$

- P is still undefined.

- Substitute this policy function into the Bellman equation in place of y_t and get the equation
$$P = R + \beta A' P A - \left(\beta A' P B + W' \right) \left[Q + \beta B' P B \right]^{-1} \left(\beta B' P A + W \right)$$

- P can be found, given some initial P_0, as the limit from iterating on the matrix Ricotti equation

$$P_{j+1} = R + \beta A' P_j A - (\beta A' P_j B + W') \left[Q + \beta B' P_j B\right]^{-1} (\beta B' P_j A + W)$$

Matrix derivatives

- The rules for taking matrix derivatives are

$$\begin{aligned} \frac{\partial x' A x}{\partial x} &= (A + A')\, x \\ \frac{\partial y' B x}{\partial y} &= Bx \\ \frac{\partial y' B x}{\partial x} &= B' y \end{aligned}$$

Finding the value matrix for Hansen's basic model

- The first step is to choose the parameter values
- From previous models, these are $\beta = .99$, $\delta = .025$, $\theta = .36$, and $A = 1.72$.
- The stationary state values are $\overline{h} = .3335$, $\overline{k} = 12.6695$, $\overline{y} = 1.2353$, and $\overline{c} = .9186$
- The resulting a matrix is

$$a = \begin{bmatrix} -0.6056 & 0.5986 & -1.3823 \\ 0.5986 & -0.5926 & 1.4048 \\ -1.3823 & 1.4048 & -6.6590 \end{bmatrix}$$

- M is

$$M = \begin{bmatrix} -1.6374 & 1.0996 & -1.0886 & 1.9361 \\ 1.0996 & -0.6056 & 0.5986 & -1.3823 \\ -1.0886 & 0.5986 & -0.5926 & 1.4048 \\ 1.9361 & -1.3823 & 1.4048 & -6.6590 \end{bmatrix}$$

Partitioning the M matrix

- $M = \begin{bmatrix} R & W' \\ W & Q \end{bmatrix}$, so using earlier M matrix gives

$$R = \begin{bmatrix} -1.6374 & 1.0996 \\ 1.0996 & -0.6056 \end{bmatrix}$$

$$Q = \begin{bmatrix} -0.5926 & 1.4048 \\ 1.4048 & -6.6590 \end{bmatrix}$$

$$W = \begin{bmatrix} -1.0886 & 0.5986 \\ 1.9361 & -1.3823 \end{bmatrix}$$

Finding the value function

- The initial P_0 is

$$P_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Use the matrix Ricotti equation and get

$$P_1 = \begin{bmatrix} -.7515 & .9987 \\ .9987 & -0.4545 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} -1.6909 & .8247 \\ .8247 & -0.1924 \end{bmatrix}$$

Results for Hansen's economy

- After 200 iterations

$$P = \begin{bmatrix} -96.3615 & .8779 \\ .8779 & -0.0259 \end{bmatrix}$$

- The matrix policy function is

$$F = \begin{bmatrix} 0.5869 & 0.9537 \\ 0.4146 & -0.0064 \end{bmatrix}.$$

Results for Hansen's economy in a stationary state

- Checking results in a stationary state

$$x = \begin{bmatrix} 1 \\ 12.6695 \end{bmatrix}$$

- Applying F gives

$$y = F * x = \begin{bmatrix} 0.5869 & 0.9537 \\ 0.4146 & -0.0064 \end{bmatrix} \begin{bmatrix} 1 \\ 12.6695 \end{bmatrix} = \begin{bmatrix} 12.6698 \\ 0.3335 \end{bmatrix}$$

- To find the x_{t+1} want

$$\begin{aligned} x_{t+1} &= Ax + By = Ax + BFx \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0.5869 & 0.9537 \\ 0.4146 & -0.0064 \end{bmatrix} x \\ &= \begin{bmatrix} 1 \\ 12.6698 \end{bmatrix} \end{aligned}$$

Adding stochastic shocks

- Add stochastic shocks through the budget constraints

$$x_{t+1} = Ax_t + By_t + C\varepsilon_{t+1}$$

where ε_t is an independent and identically distributed random variable with $E_t(\varepsilon_{t+1}) = \overrightarrow{0}$, a finite, diagonal variance matrix, Σ, and C a matrix that is $m \times n$ where m is the number of state variables and n is the length of the vector of shocks, ε_{t+1}.

Adding stochastic shocks

- Proceed as before, looking for solution to

$$E_0 \sum_{t=0}^{\infty} \beta^t z_t' M z_t,$$

subject to the linear budget constraints

$$x_{t+1} = Ax_t + By_t + C\varepsilon_{t+1}.$$

- Look for value function of the form

$$x_t' P x_t + c = \max_{\{y_s\}_{s=t}^{\infty}} E_0 \sum_{s=t}^{\infty} \beta^{s-t} z_s' M z_s,$$

- The constant is possible because of the expectations operator

Adding stochastic shocks

- The Bellman equation is

$$x_t' P x_t + c = \max_{y_t} \left\{ z_t' M z_t + \beta E_0 \left[x_{t+1}' P x_{t+1} + c \right] \right\},$$

subject to

$$x_{t+1} = Ax_t + By_t + C\varepsilon_{t+1}.$$

- This can be written as

$$\begin{aligned} x_t' P x_t + c \quad = \quad & \max_{y_t} [z_t' M z_t + \beta x_t' A' P A x_t + \beta y_t' B' P B y_t \\ & + \beta E_0 \left[\varepsilon_{t+1}' C' P C \varepsilon_{t+1} \right] + \beta c] . \end{aligned}$$

Adding stochastic shocks

- Define $G = [g_{jk}] = C' P C$

- then

$$E_t\left[\varepsilon_{t+1}'C'PC\varepsilon_{t+1}\right]=\sum_j\sum_k E_t\left[\varepsilon_{t+1}^j g_{jk}\varepsilon_{t+1}^k\right]=\sum_j g_{jj}E_t\left[\varepsilon_{t+1}^j\varepsilon_{t+1}^j\right]$$

because $E_t\left[\varepsilon_{t+1}^k\varepsilon_{t+1}^j\right]=0$, when $k\neq j$

- But $\sum_j g_{jj}=\text{trace}\,(C'PC)$
- So

$$\begin{aligned}x_t'Px_t+c &= \max_{y_t}\,[z_t'Mz_t+\beta x_t'A'PAx_t+\beta y_t'B'PBy_t\\ &\quad +\beta\text{trace}\,[C'PC\Sigma]+\beta c]\end{aligned}$$

- $c=\beta\text{trace}\,[C'PC\Sigma]\,/\,(1-\beta)$

Adding stochastic shocks

- Using this value of c, get

$$\begin{aligned}x_t'Px_t &= \max_{y_t}\,[z_t'Mz_t+\beta x_t'A'PAx_t+\beta y_t'B'PBy_t]\\ &= \max_{y_t}\,[x_t'Rx_t+y_t'Qy_t+2y_t'Wx_t+\beta x_t'A'PAx_t+\beta y_t'B'PBy_t]\end{aligned}$$

- First order conditions give

$$[Q+\beta B'PB]\,y_t=-\,[W+\beta B'PA]\,x_t$$

or

$$y_t=Fx_t=-\,[Q+\beta B'PB]^{-1}\,[W+\beta B'PA]\,x_t$$

- Exactly the same first order condition (and therefore policy matrix) as in the deterministic case
- Find time path using

$$x_{t+1}=[A+BF]\,x_t+C\varepsilon_{t+1}.$$

The basic Hansen example economy

- Agents max

$$\max\sum_{t=0}^{\infty}\beta^t\left[\ln\left(k_t^\theta h_t^{1-\theta}+(1-\delta)k_t-k_{t+1}\right)+A\ln(1-h_t)\right],$$

subject to the linear budget constrainta:

$$k_{t+1}=k_{t+1}$$

and

$$\lambda_{t+1}=(1-\gamma)+\gamma\lambda_t+\varepsilon_{t+1},$$

- Define the state variables as

$$x_t = \begin{bmatrix} 1 \\ k_t \\ \lambda_t \end{bmatrix}$$

and the controls as

$$y_t = \begin{bmatrix} k_{t+1} \\ h_t \end{bmatrix}$$

.The basic Hansen example economy

- The budget constraint can be written as

$$x_{t+1} = Ax_t + By_t + C\varepsilon_{t+1}$$

or as

$$\begin{bmatrix} 1 \\ k_{t+1} \\ \lambda_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1-\gamma & 0 & \gamma \end{bmatrix} \begin{bmatrix} 1 \\ k_t \\ \lambda_t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k_{t+1} \\ h_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \varepsilon_{t+1}$$

.The basic Hansen example economy

- The second order taylor series expansion of the objective function is (note $\widehat{a}$ parameters)

$$\begin{aligned} u(\cdot) \quad \approx \quad & \ln\left(\overline{\lambda k}^{\theta}\overline{h}^{1-\theta} - \delta\overline{k}\right) + A\ln(1-\overline{h}) \\ & + \frac{1}{\overline{c}}\left[\theta\frac{\overline{y}}{\overline{k}} + (1-\delta)\right]\left(k_t - \overline{k}\right) \\ & + \frac{\overline{y}}{\overline{c}}\left(\lambda_t - \overline{\lambda}\right) - \frac{1}{\overline{c}}\left(k_{t+1} - \overline{k}\right) \\ & + \left[(1-\theta)\frac{1}{\overline{c}}\frac{\overline{y}}{\overline{h}} - \frac{A}{1-\overline{h}}\right]\left(h_t - \overline{h}\right) \\ & + \begin{bmatrix} \left(k_t - \overline{k}\right) \\ \left(\lambda_t - \overline{\lambda}\right) \\ \left(k_{t+1} - \overline{k}\right) \\ \left(h_t - \overline{h}\right) \end{bmatrix}' \begin{bmatrix} a_{11} & \widehat{a}_{1\lambda} & a_{12} & a_{13} \\ \widehat{a}_{\lambda 1} & \widehat{a}_{\lambda\lambda} & \widehat{a}_{\lambda 2} & \widehat{a}_{\lambda 3} \\ a_{21} & \widehat{a}_{2\lambda} & a_{32} & a_{32} \\ a_{31} & \widehat{a}_{3\lambda} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \left(k_t - \overline{k}\right) \\ \left(\lambda_t - \overline{\lambda}\right) \\ \left(k_{t+1} - \overline{k}\right) \\ \left(h_t - \overline{h}\right) \end{bmatrix} \end{aligned}$$

.The basic Hansen example economy

- Get an M matrix for quadratic optimization problem

$$\max_{\{y_t\}} \sum_{t=0}^{\infty} z_{t'} M z_t,$$

subject to the budget constraints

$$x_{t+1} = Ax_t + By_t + C\varepsilon_{t+1}.$$

The 5x5 matrix M in the quadratic version of the objective function is

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\ m_{21} & a_{11} & \widehat{a}_{1\lambda} & a_{12} & a_{13} \\ m_{31} & \widehat{a}_{\lambda 1} & \widehat{a}_{\lambda\lambda} & \widehat{a}_{\lambda 2} & \widehat{a}_{\lambda 3} \\ m_{41} & a_{21} & \widehat{a}_{2\lambda} & a_{22} & a_{23} \\ m_{51} & a_{31} & \widehat{a}_{3\lambda} & a_{32} & a_{33} \end{bmatrix}$$

The m_{ij}'s are described in detail the book
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- Calibration and solution
- Only addition is $\gamma = .95$ (as before, based on estimates from US)
- Solve

$$P_{k+1} = R + \beta A' P_k A - (\beta A' P_k B + W') \left[Q + \beta B' P_k B\right]^{-1} (\beta B' P_k A + W)$$

- to find the matrix P

$$P = \begin{bmatrix} -124.0532 & 1.0657 & 15.6762 \\ 1.0657 & -0.0259 & -0.1878 \\ 15.6762 & -0.1878 & -1.9963 \end{bmatrix}$$

- and then use

$$y_t = F x_t = -\left[Q + \beta B' P B\right]^{-1} \left[W + \beta B' P A\right] x_t$$

- to find the policy function F,

$$F = \begin{bmatrix} -0.8470 & 0.9537 & 1.4340 \\ 0.1789 & -0.0064 & 0.2357 \end{bmatrix}.$$

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- Given this F and the budget constraint, get

$$\begin{aligned} \begin{bmatrix} 1 \\ k_{t+1} \\ \lambda_{t+1} \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ .05 & 0 & .95 \end{bmatrix} \begin{bmatrix} 1 \\ k_t \\ \lambda_t \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -0.8470 & 0.9537 & 1.4340 \\ 0.1789 & -0.0064 & 0.2357 \end{bmatrix} \begin{bmatrix} 1 \\ k_t \\ \lambda_t \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \varepsilon_{t+1}, \end{aligned}$$

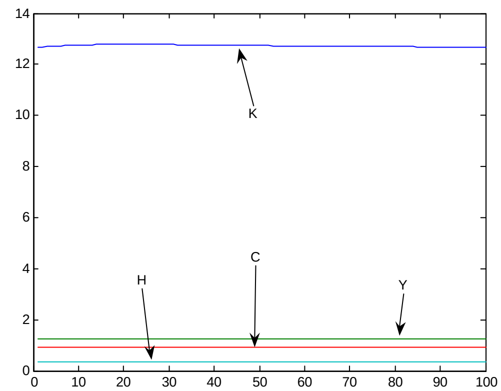

Figure 1: Impulse responses given in levels

- the laws of motion is

$$\begin{bmatrix} 1 \\ k_{t+1} \\ \lambda_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -0.8470 & 0.9537 & 1.4340 \\ .05 & 0 & .95 \end{bmatrix} \begin{bmatrix} 1 \\ k_t \\ \lambda_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \varepsilon_{t+1}.$$

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- Impulse response in levels

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- Impulse response in log differences

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- Comparing impulse response of linear quadratic to first method

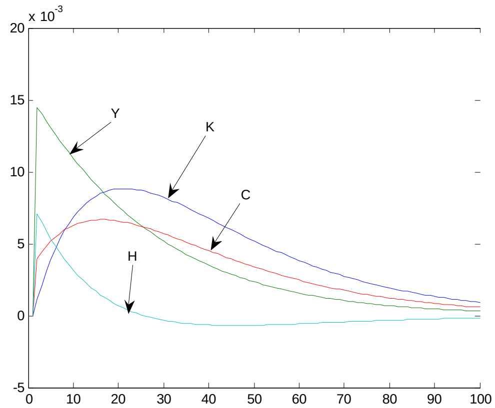

Figure 2: Responses found using linear quadratic solution method

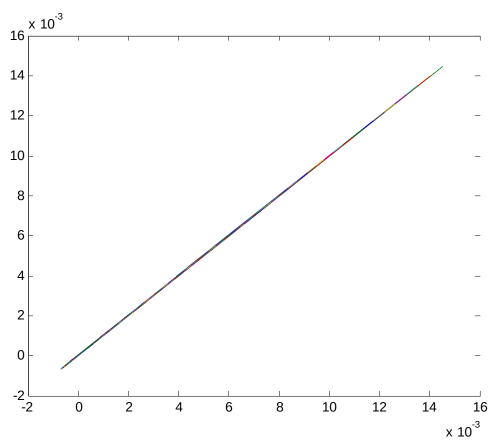

Figure 3: Comparing the two solution techniques using Hansen's model