1 Cash in advance model

Cash in advance

- Adding money to model
  - Somewhat ad hoc method
  - NOT micro foundations for why people hold money
  - we assume that they must
- Assume that one needs money to purchase consumption good
- Carry money over from pervious period (plus some possible transfers)
- velocity is constant (one cycle per period)
- Story
- I show two ways to solve the models

Model of Cooley and Hansen

- Unit mass of identical agents
- Will assume indivisible labor (not a big deal)
- Agents will need money to make consumption purchases
  - money held over from previous period
- Government can make direct lump-sum transfers or taxes of money
- Addition of money means that second welfare theorem need not hold
  - individual decisions based on
    - aggregate amount of money
    - price level

The model

- Households' maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t^i, h_t^i)$$

- where

$$u(c_t^i, h_t^i) = \ln c_t^i + \left[A \frac{\ln(1 - h_0)}{h_0} \right] h_t^i$$

- Production takes place with production function

$$y_t = \lambda_t K_t^\theta H_t^{1-\theta}$$

- Technology follows

$$\ln(\lambda_{t+1}) = \gamma \ln(\lambda_t) + \varepsilon_{t+1}$$

The model

- Comptetive factor markets imply that

$$w_t = (1-\theta)\,\lambda_t K_t^\theta H_t^{-\theta},$$

  and

$$r_t = \theta \lambda_t K_t^{\theta-1} H_t^{1-\theta}.$$

- Aggregation conditions are

$$H_t = \int_0^1 h_t^i di,$$

  and

$$K_t = \int_0^1 k_t^i di.$$

The model

- The households budget constraint is

$$c_t^i + k_{t+1}^i + \frac{m_t^i}{p_t} = w_t h_t^i + r_t k_t^i + (1-\delta)k_t^i + \frac{m_{t-1}^i + (g_t - 1)M_{t-1}}{p_t}$$

  - $\frac{m_t^i}{p_t}$ is the real value of money carried into the next period
  - $m_{t-1}^i + (g_t - 1)M_{t-1}$ is money from the previous period plus transfers (or taxes) from the government
  - g_t is the gross growth rate of money: $M_t = g_t M_{t-1}$

- The cash-in-advance constraint is

$$p_t c_t^i \leq m_{t-1}^i + (g_t - 1)\,M_{t-1}$$

  - This is an additional constraint on consumption
  - Want it to always hold (with equality)
  - Need to have gross money growth greater than discount factor, β

Normalization issues

- Models are valid close to stationary states

- Need to have stable models so that they stay near SS
- Money is a problem
  - Money is a stock
  - An increase in the growth rate imply change of level
  - No reason to return to old level
- Two methods of normalizing
  - Measure all nominal variables relative to the aggregate money stock
  - Measure all nominal variables in real terms (divided by price level)

Normalization issues

- Method of Cooley-Hansen
  - They divide all nominal variables by money stock
  - define $\widehat{p}_t = p_t/M_t$, $\widehat{m}_t^i = m_t^i/M_t$, and $M_t/M_t = 1$
  - Cash-in-advance constraint is

$$\frac{p_t}{M_t} c_t^i = \frac{m_{t-1}^i + (g_t - 1) M_{t-1}}{M_t}$$

    or

$$\begin{aligned} \widehat{p}_t c_t^i &= \frac{m_{t-1}^i + (g_t - 1) M_{t-1}}{g_t M_{t-1}} \\ \widehat{p}_t c_t^i &= \frac{\widehat{m}_{t-1}^i + (g_t - 1)}{g_t} \end{aligned}$$

  - Budget constraint is

$$c_t^i + k_{t+1}^i + \frac{\widehat{m}_t^i}{\widehat{p}_t} = w_t h_t^i + r_t k_t^i + (1 - \delta) k_t^i + \frac{\widehat{m}_{t-1}^i + (g_t - 1)}{g_t \widehat{p}_t}$$

Normalization issues

- Real balance method
  - Divide all nominal variables by the price level
  - They usually show up in real equations in this form anyway
  - A family's real balances are

$$\overline{m/p} = \frac{m_t^i}{p_t},$$

    and the economy real balances are

$$\overline{M/p} = \frac{M_t}{p_t}.$$

- Some care needs to be taken with the lagged money variables. In a stationary state, $\overline{g} = \overline{\pi}$. In the stationary state

$$\frac{m_{t-1}^i}{p_t} = \frac{m_{t-1}^i}{\pi_t p_{t-1}} = \frac{\overline{m/p}}{\overline{\pi}} = \frac{\overline{m/p}}{\overline{g}}.$$

Normalization issues

- Cash-in-advance constraint is

$$c_t^i = \frac{m_{t-1}^i}{P_t}\frac{P_{t-1}}{P_{t-1}} + (g_t - 1)\frac{M_{t-1}}{P_t}\frac{P_{t-1}}{P_{t-1}}$$

$$c_t^i = \frac{m_{t-1}^i}{P_{t-1}}\frac{1}{\pi_t} + (g_t - 1)\frac{M_{t-1}}{P_{t-1}}\frac{1}{\pi_t}$$

- The flow budget constraint (after removing the cash in advance constraint) is

$$k_{t+1}^i + \frac{m_t^i}{P_t} = w_t h_t^i + r_t k_t^i + (1-\delta)k_t^i$$

- Will have variables P_t and M_t that could have a unit root
  - not a real problem because they always appear together
    - and they are co-integrated
  - real variables of model do not have unit roots
- Go back to Cooley-Hansen's way

Full model

- Households max

$$\max E_0 \sum_{t=0}^{\infty}\left(\beta^t \ln c_t^i + \left[A\frac{\ln(1-h_0)}{h_0}\right] h_t^i\right)$$

subject to the budget constraints

$$c_t^i = \frac{\widehat{m}_{t-1}^i + (g_t - 1)}{g_t \widehat{p}_t}$$

$$\begin{aligned} c_t^i + k_{t+1}^i + \frac{\widehat{m}_t^i}{\widehat{p}_t} &= \left((1-\theta)\lambda_t K_t^{\theta} H_t^{-\theta}\right) h_t^i + \left(\theta\lambda_t K_t^{\theta-1} H_t^{1-\theta}\right) k_t^i \\ &\quad +(1-\delta)k_t^i + \frac{\widehat{m}_{t-1}^i + (g_t - 1)}{g_t \widehat{p}_t} \end{aligned}$$

- The law of motion for the stochastic shock

$$\ln \lambda_{t+1} = \gamma \ln \lambda_t + \varepsilon^{\lambda}_{t+1}$$

Full model

- The growth rate for money that is either a stationary state rule,

$$g_t = \overline{g}$$

  or a stochastic rule

$$\ln g_{t+1} = (1-\pi)\ln \overline{g} + \pi \ln g_t + \varepsilon^{g}_{t+1}$$

- The aggregation conditions for an equilibrium are

$$\begin{aligned} K_t &= k^i_t \\ H_t &= h^i_t \\ C_t &= c^i_t \end{aligned}$$

  and

$$\widehat{M}_t = \widehat{m}^i_t = 1$$

Solving the model

- The first order conditions and constraints for households

$$\begin{aligned} \frac{1}{\beta} &= E_t \frac{w_t}{w_{t+1}} \left[(1-\delta) + r_{t+1}\right] \\ \frac{B\overline{g}}{w_t \widehat{p}_t} &= -\beta E_t \frac{1}{\widehat{p}_{t+1} c^i_{t+1}} \\ \widehat{p}_t c^i_t &= \frac{\widehat{m}^i_{t-1} + g_t - 1}{g_t} \\ k^i_{t+1} + \frac{\widehat{m}^i_t}{\widehat{p}_t} &= (1-\delta)\, k^i_t + w_t h^i_t + r_t k^i_t \end{aligned}$$

- Factor market conditions

$$w_t = (1-\theta)\,\lambda_t \left[\frac{K_t}{H_t}\right]^{\theta} \text{ and } r_t = \theta \lambda_t \left[\frac{K_t}{H_t}\right]^{\theta-1}$$

- Equilibrium conditions

$$C_t = c^i_t, \quad H_t = h^i_t, \quad K_{t+1} = k^i_{t+1}, \text{ and } \widehat{M}_t = \widehat{m}^i_t = 1$$

Stationary state

- The equations for finding the stationary state

$$
\begin{aligned}
\frac{1}{\beta} &= (1-\delta)+\overline{r} \\
\frac{B}{\overline{w}} &= -\frac{\beta}{\overline{g}\overline{C}} \\
\widehat{p}\overline{C} &= 1 \\
\frac{1}{\widehat{p}} &= (\overline{r}-\delta)\,\overline{K}+\overline{w}\overline{H} \\
\overline{w} &= (1-\theta)\left[\frac{\overline{K}}{\overline{H}}\right]^{\theta} \\
\overline{r} &= \theta\left[\frac{\overline{K}}{\overline{H}}\right]^{\theta-1}
\end{aligned}
$$

Stationary state

- Solving the equations of the stationary state give

$$
\begin{aligned}
\overline{r} &= \frac{1}{\beta}-(1-\delta) \\
\overline{w} &= (1-\theta)\left[\frac{\overline{K}}{\overline{H}}\right]^{\theta}=(1-\theta)\left[\frac{\overline{r}}{\theta}\right]^{\frac{\theta}{\theta-1}} \\
\overline{C} &= -\frac{\beta\overline{w}}{\overline{g}B} \\
\widehat{p} &= \frac{1}{\overline{C}} \\
\overline{K} &= \frac{\overline{C}}{\frac{\overline{r}}{\theta}-\delta} \\
\overline{H} &= \left(\frac{\overline{r}}{\theta}\right)^{\frac{1}{1-\theta}}\overline{K} \\
\overline{Y} &= \overline{C}+\delta\overline{K}
\end{aligned}
$$

Stationary state

- Parameter values are $\beta=.99$, $\delta=.025$, $\theta=.36$, $A=1.72$, and $h_0=.583$, so $B=-2.5805$
- These give stationary state values of

$variable$	$value\ in\ s.s.$
$\overline{r}$	$.03\,51$
$\overline{w}$	$2.\,370\,6$
$\overline{C}$	$\frac{0.9095}{\overline{g}}$
$\widehat{p}$	$1.\,0995\overline{g}$
$\overline{K}$	$\frac{12.\,544}{\overline{g}}$
$\overline{H}$	$\frac{0.3302}{\overline{g}}$
$\overline{Y}$	$\frac{1.\,2231}{\overline{g}}$

- Notice how the growth rate of money affects real variables
- It is also possible to calculate the welfare loss from inflation:utility $=\frac{\ln\left(\frac{0.9095}{\overline{g}}\right)-2.5805\frac{0.3302}{\overline{g}}}{1-.99}$ $= -100\ln g - \frac{85.\,208}{g} - 9.\,486$

Solving the dynamic model: version 1

- Cooley and Hansen used linear quadratic method
- Problem: there are two economy wide variables, K_t and $\widehat{p}_t$
- These do not come directly from individual maximization problems
- Come from aggregation or equilibrium conditions
- Individual maximization problems do depend on these
- (we will also want to remove labor (both individual and aggregate) from model

Solving the dynamic model: version 1

- How to proceed
- eliminate consumption from optimization problem using c-i-a constraint

$$\max E_0 \sum_{t=0}^{\infty} \left(\beta^t \ln \left[\frac{\widehat{m}_{t-1}^i + (g_t - 1)}{g_t \widehat{p}_t} \right] + \left[A \frac{\ln(1-h_0)}{h_0} \right] h_t^i \right)$$

- Using remaining budget constaint

$$k_{t+1}^i + \frac{\widehat{m}_t^i}{\widehat{p}_t} = \left((1-\theta)\, \lambda_t K_t^\theta H_t^{-\theta} \right) h_t^i + \left(\theta \lambda_t K_t^{\theta-1} H_t^{1-\theta} \right) k_t^i + (1-\delta) k_t^i$$

and simplify to get

$$k_{t+1}^i - (1-\delta) k_t^i + \frac{\widehat{m}_t^i}{\widehat{p}_t} = \left(\lambda_t K_t^\theta H_t^{1-\theta} \right) \left[(1-\theta) \frac{h_t^i}{H_t} + \theta \frac{k_t^i}{K_t} \right]$$

Solving the dynamic model: version 1

- Sum across households to get

$$K_{t+1} + \frac{1}{\widehat{p}_t} = \lambda_t K_t^{\theta} H_t^{1-\theta} + (1-\delta) K_t$$

which can be solved for aggregate labor as

$$H_t = \left[\frac{K_{t+1} - (1-\delta)K_t + \frac{1}{\widehat{p}_t}}{\lambda_t K_t^{\theta}} \right]^{\frac{1}{1-\theta}}$$

- Individual labor is then

$$h_t^i = \frac{k_{t+1}^i - (1-\delta)k_t^i + \frac{\widehat{m}_t^i}{\widehat{p}_t} - \theta \left[K_{t+1} - (1-\delta)K_t + \frac{1}{\widehat{p}_t} \right] \frac{k_t^i}{K_t}}{(1-\theta) \left[K_{t+1} - (1-\delta)K_t + \frac{1}{\widehat{p}_t} \right]^{-\frac{\theta}{1-\theta}} \left[\lambda_t K_t^{\theta} \right]^{\frac{1}{1-\theta}}}$$

Solving the dynamic model: version 1

- Put all this into the objective function

$$\max_{k_{t+1}^i, \widehat{m}_t^i} E_0 \sum_{t=0}^{\infty} \left(\beta^t \ln \left[\frac{\widehat{m}_{t-1}^i + (g_t - 1)}{g_t \widehat{p}_t} \right] + \left[A \frac{\ln(1-h_0)}{h_0} \right] \times \left[\frac{k_{t+1}^i - (1-\delta)k_t^i + \frac{\widehat{m}_t^i}{\widehat{p}_t} - \theta \left[K_{t+1} - (1-\delta)K_t + \frac{1}{\widehat{p}_t} \right] \frac{k_t^i}{K_t}}{(1-\theta) \left[K_{t+1} - (1-\delta)K_t + \frac{1}{\widehat{p}_t} \right]^{-\frac{\theta}{1-\theta}} \left[\lambda_t K_t^{\theta} \right]^{\frac{1}{1-\theta}}} \right] \right)$$

subject to the budget constraints

$$k_{t+1}^i = k_{t+1}^i$$

$$\widehat{m}_t^i = \widehat{m}_t^i$$

$$\ln(\lambda_{t+1}) = \gamma \ln(\lambda_t) + \varepsilon_{t+1}^{\lambda}$$

$$\ln g_{t+1} = (1-\pi)\,\overline{g} + \pi \ln g_t + \varepsilon_{t+1}^{g}$$

Solving the dynamic model: version 1

- State variables: $x_t^i = \begin{bmatrix} 1 & \lambda_t & k_t^i & \widehat{m}_{t-1}^i & g_t & K_t \end{bmatrix}'$
- Control variables: $y_t^i = \begin{bmatrix} k_{t+1}^i & \widehat{m}_t^i \end{bmatrix}'$
- Economy wide variables $Z_t = \begin{bmatrix} K_{t+1} & \widehat{p}_t \end{bmatrix}'$
- Write the linear quadratic objective function as

$$\begin{bmatrix} x_t' & y_t' & Z_t' \end{bmatrix} Q \begin{bmatrix} x_t \\ y_t \\ Z_t \end{bmatrix}$$

- Given this objective function, we want to solve a Bellmans equation of the form

$$x_t' P x_t = \max_{y_t} \left[\begin{bmatrix} x_t' & y_t' & Z_t' \end{bmatrix} Q \begin{bmatrix} x_t \\ y_t \\ Z_t \end{bmatrix} + \beta E_0 \left[x_{t+1}' P x_{t+1} \right] \right]$$

subject to the budget constraints

$$x_{t+1} = Ax_t + By_t + CZ_t + D\varepsilon_{t+1}$$

Solving the dynamic model: version 1

- rewrite the matrix Q as

$$Q = \begin{bmatrix} R & W' & X' \\ W & T & N' \\ X & N & S \end{bmatrix},$$

- write

$$\begin{bmatrix} x_t' & y_t' & Z_t' \end{bmatrix} Q \begin{bmatrix} x_t \\ y_t \\ Z_t \end{bmatrix}$$

as

$$x_t' R x_t + y_t' T y_t + Z_t' S Z_t + 2y_t' W x_t + 2Z_t' X x_t + 2Z_t' N y_t.$$

- The last part of the Bellmans equation can be written as

$$\begin{aligned} & \beta E_0 \left[x_{t+1}' P x_{t+1} \right] \\ = \; & \beta E_0 \left[(Ax_t + By_t + CZ_t + D\varepsilon_{t+1})' P (Ax_t + By_t + CZ_t + D\varepsilon_{t+1}) \right] \end{aligned}$$

Solving the dynamic model: version 1

- First order condition are

$$0 = Ty_t + Wx_t + N'Z_t + \beta \left[B'PAx_t + B'PBy_t + B'PCZ_t \right]$$

or

$$(T + \beta B'PB) y_t = -(W + \beta B'PA) x_t - (N + \beta B'PC) Z_t$$

- When $(T + \beta B'PB)$ is invertible, the *linear* policy function is

$$\begin{aligned} y_t = \; & -(T + \beta B'PB)^{-1} (W + \beta B'PA) x_t \\ & -(T + \beta B'PB)^{-1} (N + \beta B'PC) Z_t \end{aligned}$$

- which we can write as
$$y_t = F_1 x_t + F_2 Z_t,$$
with
$$\begin{aligned} F_1 &= -\left(T + \beta B' P B\right)^{-1} \left(W + \beta B' P A\right) \\ F_2 &= -\left(T + \beta B' P B\right)^{-1} \left(N + \beta B' P C\right) \end{aligned}$$

Solving the dynamic model: version 1

- The value function P that we want fulfills

$$\begin{aligned} & x_t' P x_t \\ = & \begin{bmatrix} x_t' & \left(F_1 x_t + F_2 Z_t\right)' & Z_t' \end{bmatrix} Q \begin{bmatrix} x_t \\ F_1 x_t + F_2 Z_t \\ Z_t \end{bmatrix} \\ & +\beta \left[\left(A + BF_1\right) x_t + \left(BF_2 + C\right) Z_t\right]' P \left[\left(A + BF_1\right) x_t + \left(BF_2 + C\right) Z_t\right] \end{aligned}$$

- Unfortunately, the Z_t variables are still a problem

Solving the dynamic model: version 1

- Handling the economy wide variables
- We can aggregate (integrate over) the controls to get

$$\int_0^1 y_t^i di = \begin{bmatrix} \int_0^1 k_{t+1}^i di \\ \int_0^1 \widehat{m}_t^i di \end{bmatrix} = \begin{bmatrix} K_{t+1} \\ 1 \end{bmatrix}$$

- An aggregated version of the policy function is
$$\int_0^1 y_t^i di = F_1 \int_0^1 x_t^i di + F_2 Z_t$$
or
$$\begin{bmatrix} K_{t+1} \\ 1 \end{bmatrix} = F_1 \int_0^1 x_{t+1}^i di + F_2 Z_t$$

Solving the dynamic model: version 1

- Since
$$x_t^i = \begin{bmatrix} 1 & \lambda_t & k_t^i & \widehat{m}_{t-1}^i & g_t & K_t \end{bmatrix}',$$
- The integral of this vector is
$$\widehat{x}_t = \int_0^1 x_{t+1}^i di = \begin{bmatrix} 1 & \lambda_t & K_t & 1 & g_t & K_t \end{bmatrix},$$

- we can construct a matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

- So that $\widehat{x}_t = Gx_t^i$, for all i

Solving the dynamic model: version 1

- The aggregate version of the policy function is

$$\begin{bmatrix} K_{t+1} \\ 1 \end{bmatrix} = F_1 G x_t^i + F_2 \begin{bmatrix} K_{t+1} \\ \widehat{p}_t \end{bmatrix}$$

- This equation can be solved for the vector $\begin{bmatrix} K_{t+1} \\ \widehat{p}_t \end{bmatrix}$ as

$$\begin{bmatrix} K_{t+1} \\ \widehat{p}_t \end{bmatrix} = F_2^{-1} \begin{bmatrix} K_{t+1} \\ 1 \end{bmatrix} - F_2^{-1} F_1 G x_t^i$$

or as

$$\begin{bmatrix} K_{t+1} \\ \widehat{p}_t \end{bmatrix} = J \begin{bmatrix} K_{t+1} \\ 1 \end{bmatrix} + H x_t^i,$$

Solving the dynamic model: version 1

- Recalling that the first element of x_t^i is always 1, one can find a function of the form

$$Z_t = F_3 x_t^i$$

- Here

$$F_3 = \left[\begin{matrix} \frac{H_{11}+J_{12}}{1-J_{11}} & \frac{H_{12}}{1-J_{11}} \\ H_{21}+J_{22}+\frac{J_{21}(H_{11}+J_{12})}{1-J_{11}} & H_{22}+\frac{J_{21}H_{12}}{1-J_{11}} \end{matrix} \right.$$
$$\begin{matrix} \frac{H_{13}}{1-J_{11}} & \frac{H_{14}}{1-J_{11}} \\ H_{23}+\frac{J_{21}H_{13}}{1-J_{11}} & H_{24}+\frac{J_{21}H_{14}}{1-J_{11}} \end{matrix}$$
$$\left. \begin{matrix} \frac{H_{15}}{1-J_{11}} & \frac{H_{16}}{1-J_{11}} \\ H_{25}+\frac{J_{21}H_{15}}{1-J_{11}} & H_{26}+\frac{J_{21}H_{16}}{1-J_{11}} \end{matrix} \right].$$

Solving the dynamic model: version 1

- Bellman equation is

$$\begin{aligned}
&P \\
= &\begin{bmatrix} I_x & F_1' + F_3' F_2' & F_3' \end{bmatrix} Q \begin{bmatrix} I_x \\ F_1 + F_2 F_3 \\ F_3 \end{bmatrix} \\
&+\beta \left[(A + BF_1) + (BF_2 + C) F_3 \right]' P \left[(A + BF_1) + (BF_2 + C) F_3 \right]
\end{aligned}$$

- To solve, choose P_0
- Find F_1^0, F_2^0 and using these find F_3^0
- Use these along with P_0 in the above equation to find P_1
- Repeat until conversion is close enough

.Alternative method for solving

- Log-linearization of the model
  - First order conditions
  - budget constraints
  - market equilibrium conditions (competitive or not)
  - Aggregation and other equilibrium conditions
- Economy wide variables are not, in general, a problem
  - optimization already done
  - model usually in aggregate variables

.Cash in advance Model

- First order conditions

$$\begin{aligned}
\frac{1}{\beta} &= E_t \frac{w_t}{w_{t+1}} \left[(1 - \delta) + r_{t+1} \right], \\
\frac{B \overline{g}}{w_t \widehat{p}_t} &= -\beta E_t \frac{1}{\widehat{p}_{t+1} c_{t+1}^i},
\end{aligned}$$

- the cash in advance constraint

$$\widehat{p}_t c_t^i = \frac{\widehat{m}_{t-1}^i + g_t - 1}{g_t},$$

- the flow budget constraint

$$k_{t+1}^i + \frac{\widehat{m}_t^i}{\widehat{p}_t} = (1 - \delta) k_t^i + w_t h_t^i + r_t k_t^i.$$

Cash in advance Model

- Factor market conditions

$$w_t = (1-\theta)\,\lambda_t \left[\frac{K_t}{H_t}\right]^{\theta},$$

  and

$$r_t = \theta\lambda_t \left[\frac{K_t}{H_t}\right]^{\theta-1}.$$

- Equilibrium and aggregation conditions are

$$\begin{aligned} C_t &= c_t^i \qquad H_t = h_t^i, \\ K_{t+1} &= k_{t+1}^i \qquad \widehat{M}_t = \widehat{m}_t^i = 1. \end{aligned}$$

- Stochastic processes

$$\ln\lambda_{t+1} = \gamma\ln\lambda_t + \varepsilon_t^{\lambda},$$

  and

$$\ln g_{t+1} = (1-\pi)\ln\overline{g} + \pi\ln g_t + \varepsilon_{t+1}^{g}.$$

Log-linear version of model

- The log-linear version of the first order conditions are

$$-\widetilde{w}_t = \beta E_t\left[\overline{r}\left(\widetilde{r}_{t+1} - \widetilde{w}_{t+1}\right) - (1-\delta)\,\widetilde{w}_{t+1}\right]$$

  and

$$-\frac{B}{\overline{pw}}\left[\widetilde{p}_t + \widetilde{w}_t\right] = \beta E_t\left[\frac{1}{\overline{g}}\widetilde{g}_{t+1}\right]$$

  having used the cash in advance constraint in the form,

$$g_t\widehat{p}_t c_t^i = \widehat{m}_{t-1}^i + g_t - 1$$

- The flow budget constraint is

$$\overline{k}\widetilde{k}_{t+1} + \frac{\overline{m}}{\overline{p}}\left[\widetilde{m}_t - \widetilde{p}_t\right] = \overline{w}\overline{h}\left[\widetilde{w}_t + \widetilde{h}_t\right] + \overline{r}\overline{k}\left[\widetilde{r}_t + \widetilde{k}_t\right] + (1-\delta)\overline{k}\widetilde{k}_t$$

Log-linear version of model

- Factor market condtions are

$$\overline{r}\widetilde{r}_t = \overline{K}^{\theta-1}\overline{H}^{1-\theta}\left[\widetilde{\lambda}_t + (\theta-1)\left[\widetilde{K}_t - \widetilde{H}_t\right]\right]$$

  and

$$\overline{w}\widetilde{w}_t = \overline{K}^{\theta}\overline{H}^{-\theta}\left[\widetilde{\lambda}_t + \theta\left[\widetilde{K}_t - \widetilde{H}_t\right]\right]$$

- The stochastic processes are

$$\widetilde{\lambda}_{t+1} = \gamma\widetilde{\lambda}_t + \varepsilon^{\lambda}_{t+1}$$

and

$$\widetilde{g}_{t+1} = \pi\widetilde{g}_t + \varepsilon^{g}_{t+1}$$

.Getting rid of an annoying expectations

- One can remove the expectations from

$$-\frac{B}{\overline{pw}}\left[\widetilde{p}_t + \widetilde{w}_t\right] = \beta E_t\left[\frac{1}{\overline{g}}\widetilde{g}_{t+1}\right]$$

- by using the process for money growth,

$$\widetilde{g}_{t+1} = \pi\widetilde{g}_t + \varepsilon^{g}_{t+1}$$

- Since the expectation of the error is zero, one can eliminate the expectations operator, and get

$$-\frac{B}{\overline{pw}}\left[\widetilde{p}_t + \widetilde{w}_t\right] = \frac{\beta\pi}{\overline{g}}\widetilde{g}_t$$

.The full model

- The equations without expectations are

$$\begin{aligned}
0 &= \overline{K}\widetilde{K}_{t+1} - \frac{1}{\overline{p}}\widetilde{p}_t - \overline{w}\overline{H}\widetilde{w}_t - \overline{w}\overline{H}\widetilde{H}_t - \overline{r}\overline{K}\widetilde{r}_t - \overline{r}\overline{K}\widetilde{K}_t - (1-\delta)\overline{K}\widetilde{K}_t,\\
0 &= \widetilde{r}_t - \widetilde{\lambda}_t - (\theta-1)\widetilde{K}_t + (\theta-1)\widetilde{H}_t,\\
0 &= \widetilde{w}_t - \widetilde{\lambda}_t - \theta\widetilde{K}_t + \theta\widetilde{H}_t,\\
0 &= \widetilde{p}_t + \widetilde{w}_t - \pi\widetilde{g}_t
\end{aligned}$$

- one equation in expectations

$$0 = \widetilde{w}_t + \beta\overline{r}E_t\widetilde{r}_{t+1} - E_t\widetilde{w}_{t+1},$$

- two stochastic processes for the shocks to technology and money growth,

$$\begin{aligned}
\widetilde{\lambda}_{t+1} &= \gamma\widetilde{\lambda}_t + \varepsilon^{\lambda}_{t+1},\\
\widetilde{g}_{t+1} &= \pi\widetilde{g}_t + \varepsilon^{g}_{t+1}.
\end{aligned}$$

.Solving the model

- The model can be written as

$$
\begin{aligned}
0 &= Ax_t + Bx_{t-1} + Cy_t + Dz_t, \\
0 &= E_t\left[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t\right], \\
z_{t+1} &= Nz_t + \varepsilon_{t+1},
\end{aligned}
$$

where $x_t = \left[\widetilde{K}_{t+1}\right]$, $y_t = \begin{bmatrix} \widetilde{r}_t \\ \widetilde{w}_t \\ \widetilde{H}_t \\ \widetilde{p}_t \end{bmatrix}$,and $z_t = \begin{bmatrix} \widetilde{\lambda}_t \\ \widetilde{g}_t \end{bmatrix}$,

.The matrices A to N are

$$A = \begin{bmatrix} \overline{K} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -\left(\overline{r} + 1 - \delta\right)\overline{K} \\ (1-\theta) \\ -\theta \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} -\overline{r}\overline{K} & -\overline{w}\overline{H} & -\overline{w}\overline{H} & -\frac{1}{\overline{p}} \\ 1 & 0 & (\theta-1) & 0 \\ 0 & 1 & \theta & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}$$

.

$$D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & \pi \end{bmatrix}$$

$$F = [0] \qquad G = [0] \qquad H = [0]$$

$$J = \begin{bmatrix} \beta\overline{r} & -1 & 0 & 0 \end{bmatrix}$$

$$K = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 0 & 0 \end{bmatrix} \qquad M = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$N = \begin{bmatrix} \gamma & 0 \\ 0 & \pi \end{bmatrix}$$

.Solution of model

- We look for a solution of the form

$$x_{t+1} = Px_t + Qz_t$$

and

$$y_t = Rx_t + Sz_t$$

- Where

$$(F - JC^{-1}A)P^2 - (JC^{-1}B - G + KC^{-1}A)P - KC^{-1}B + H = 0,$$

and that

$$R = -C^{-1}(AP + B),$$

$$\begin{aligned} & vec(Q) \\ = \; & \left(N' \otimes (F - JC^{-1}A) + I_k \otimes (FP + G + JR - KC^{-1}A)\right)^{-1} \\ & \times vec\left(\left(JC^{-1}D - L\right) N + KC^{-1}D - M\right), \end{aligned}$$

and

$$S = -C^{-1}(AQ + D).$$

.The solution matrices are

-

$$P = [0.9418]$$

$$Q = \begin{bmatrix} 0.1552 & 0.0271 \end{bmatrix}$$

$$R = \begin{bmatrix} -0.9450 \\ 0.5316 \\ -0.4766 \\ -0.5316 \end{bmatrix}$$

$$S = \begin{bmatrix} 1.9418 & -0.0555 \\ 0.4703 & 0.0312 \\ 1.4715 & -0.0867 \\ -0.4703 & 0.4488 \end{bmatrix}$$

.Variances

- How adding money shocks affect variances

Variable	$\sigma_\lambda = .0036$ $\sigma_g = 0$	$\sigma_\lambda = .0036$ $\sigma_g = .01$	$\sigma_\lambda = .0036$ $\sigma_g = .02$
$\widetilde{Y}$	0.0176	0.0176	0.0178
$\widetilde{C}$	0.0098	0.0119	0.0168
$\widetilde{I}$	0.0478	0.0496	0.0535
$\widetilde{K}$	0.0130	0.0129	0.0130
$\widetilde{r}$	0.0147	0.0147	0.0148
$\widetilde{w}$	0.0098	0.0098	0.0098
$\widetilde{H}$	0.0110	0.0110	0.0112
$\widetilde{p}$	0.0098	0.0109	0.0138

.Correlations with output

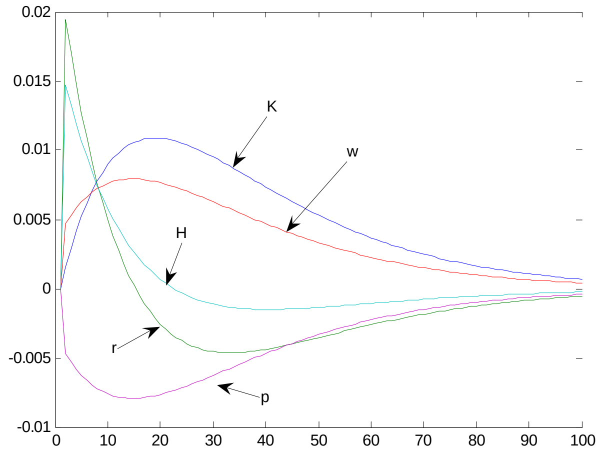


Figure 1: Response of Cooley-Hansen model to technology shock

- More money shocks reduce correlations with output

Variable	$\sigma_\lambda = .0036$ $\sigma_g = 0$	$\sigma_\lambda = .0036$ $\sigma_g = .01$	$\sigma_\lambda = .0036$ $\sigma_g = .02$
$\widetilde{Y}$	1.0000	1.0000	1.0000
$\widetilde{C}$	0.8234	0.6666	0.5094
$\widetilde{I}$	0.9472	0.9060	0.8030
$\widetilde{K}$	0.6166	0.6106	0.5966
$\widetilde{r}$	0.7149	0.7173	0.7161
$\widetilde{w}$	0.8234	0.8186	0.8045
$\widetilde{H}$	0.8753	0.8758	0.8715
$\widetilde{p}$	−0.8234	−0.7291	−0.5993

Impulse response to technology shock
Response of Hansen model (no money) to tech shock
Response of Cooley-Hansen to money growth shock
Comments

- Note that variances and impulse response functions do not depend on the level of stationary state inflation
- Look at the first row of the A, B, C, D matrices
- All elements are divided by $\overline{g}$
- The relative values of this equation do not change with $\overline{g}$
- So the dynamic model does not change with $\overline{g}$

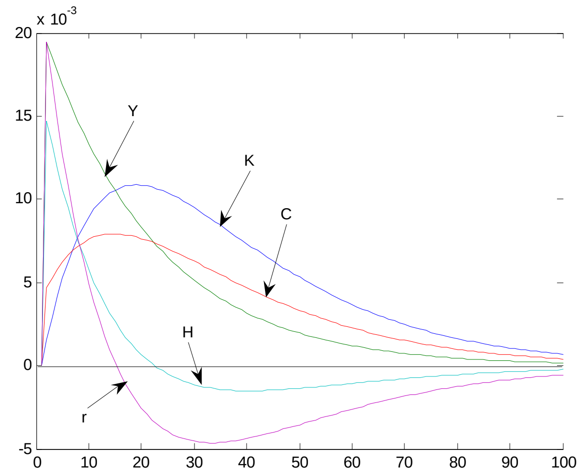


Figure 2: Response of Hansen's model to technology shock

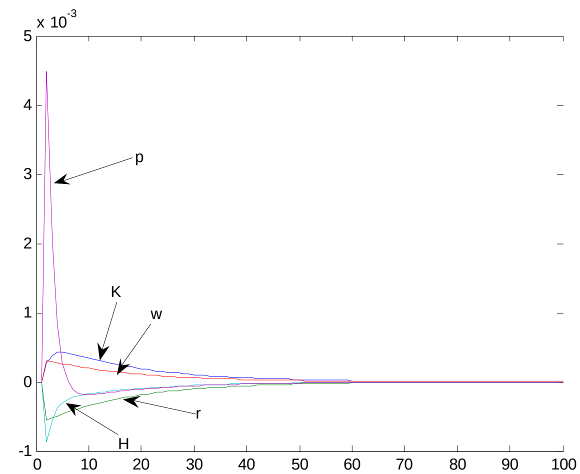


Figure 3: Response of Cooley-Hansen model to money growth shock

Seigniorage

- Alternative method of adding money to the economy
- Government consumes some goods
- Pays for these goods by issuing new money
- Budget constraint of the government is

$$g_t = \widehat{g}_t \overline{g} = \frac{M_t - M_{t-1}}{p_t}$$

  with the stochastic process

$$\ln \widehat{g}_t = \pi \ln \widehat{g}_{t-1} + \varepsilon_t^g$$

- Money issued depends on the real purchases of the government

Seigniorage

- Normalize by money stock at date t
- Government budget constraint becomes

$$g_t = \widehat{g}_t \overline{g} = \frac{\frac{M_t}{M_t} - \frac{M_{t-1}}{M_t}}{\frac{p_t}{M_t}} = \frac{1 - \frac{1}{\varphi_t}}{\widehat{p}_t}$$

- Notation: Now $\varphi_t = M_t / M_{t-1}$
  - It is the gross growth rate of money (NOT g_t)

Seigniorage

- Rest of model: household optimization problem

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t^i, 1 - h_t^i),$$

- subject to the sequence of cash in advance constraints,

$$\widehat{p}_t c_t^i \leq \frac{\widehat{m}_{t-1}^i}{\varphi_t},$$

- the sequence of family real budget constraints,

$$k_{t+1}^i + \frac{\widehat{m}_t^i}{\widehat{p}_t} = w_t h_t^i + r_t k_t^i + (1 - \delta) k_t^i.$$

Seigniorage

- the economy wide cash in advance constraints (at equality) are

$$p_tC_t + p_tg_t = p_tC_t + p_t\widehat{g}_t\overline{g} = M_t,$$

or

$$\widehat{p}_tC_t + \widehat{p}_t\widehat{g}_t\overline{g} = 1,$$

- The cash in advance for the households is

$$p_tC_t = M_{t-1},$$

- dividing both sides of this equation by M_t,

$$\widehat{p}_tC_t = \frac{1}{\varphi_t},$$

- The real budget constraint for the economy is

$$C_t + K_{t+1} + \widehat{g}_t\overline{g} = w_tH_t + r_tK_t + (1-\delta)K_t,$$

Seigniorage

- Competitive factor markets imply that

$$r_t = \theta\lambda_tK_t^{\theta-1}H_t^{1-\theta},$$

and

$$w_t = (1-\theta)\,\lambda_tK_t^{\theta}H_t^{-\theta},$$

Seigniorage

- First order conditions are

$$\frac{1}{w_t} = \beta E_t\left[\frac{r_{t+1}+1-\delta}{w_{t+1}}\right]$$

- and

$$-\frac{B}{\widehat{p}_tw_t} = \frac{\beta}{\widehat{m}_t}.$$

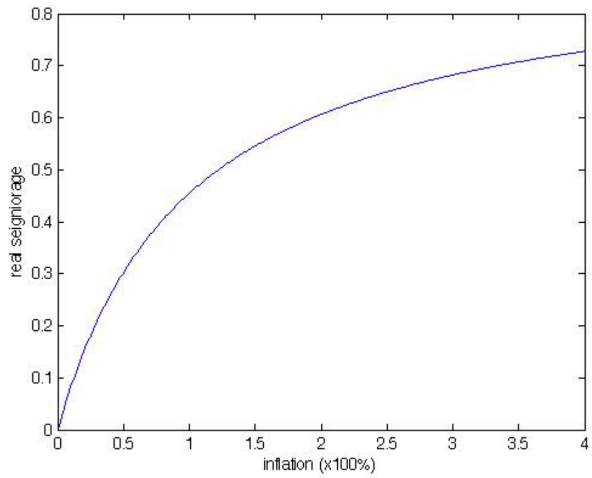
Seigniorage: Stationary state

- From FOCs

$$\frac{1}{\beta} = \overline{r} + (1-\delta)$$

and

$$-\frac{\beta\overline{w}}{B} = \frac{\widehat{m}}{\widehat{p}}$$



- From factor market

$$\overline{w} = (1-\theta)\left[\frac{\theta}{\frac{1}{\beta}-(1-\delta)}\right]^{\frac{\theta}{1-\theta}}$$

- From government budgetr constraint

$$\overline{g}\widehat{p} = 1 - \frac{1}{\varphi}$$

- Some algebra gives

$$\varphi = \frac{\beta\overline{w}}{B\overline{g}+\beta\overline{w}}$$

Seigniorage

- Bailey curve (example economy)

Seigniorage: log-linear version

- Model is

$$\begin{aligned}
0 &= \widetilde{w}_t + \overline{r}\beta E_t\widetilde{r}_{t+1} - E_t\widetilde{w}_{t+1}, \\
0 &= -\widetilde{w}_t + \widetilde{p}_t, \\
0 &= \overline{pg}\left[\widetilde{p}_t + \widetilde{g}_t\right] - \frac{1}{\overline{\varphi}}\widetilde{\varphi}_t, \\
0 &= \overline{K}\widetilde{K}_{t+1} - \frac{1}{\overline{p}}\widetilde{p}_t - \overline{w}\overline{H}\left[\widetilde{w}_t + \widetilde{H}_t\right] - \overline{r}\overline{K}\left[\widetilde{r}_t + \widetilde{K}_t\right] - (1-\delta)\overline{K}\widetilde{K}_t, \\
0 &= \widetilde{r}_t - \widetilde{\lambda}_t - (\theta-1)\widetilde{K}_t - (1-\theta)\widetilde{H}_t, \\
0 &= \widetilde{w}_t - \widetilde{\lambda}_t - \theta\widetilde{K}_t + \theta\widetilde{H}_t.
\end{aligned}$$

- Plus the stochastic processes for technology and government expenditures

Seigniorage: log-linear version

- Define variables as $x_t = \left[\widetilde{K}_{t+1}\right]$, $y_t = \begin{bmatrix} \widetilde{r}_t \\ \widetilde{w}_t \\ \widetilde{p}_t \\ \widetilde{\varphi}_t \\ \widetilde{H}_t \end{bmatrix}$, and $z_t = \begin{bmatrix} \widetilde{g}_t \\ \widetilde{\lambda}_t \end{bmatrix}$,
- write the model as we did earlier
-

$$\begin{aligned} 0 &= Ax_t + Bx_{t-1} + Cy_t + Dz_t, \\ 0 &= E_t \left[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t \right], \\ z_{t+1} &= Nz_t + \varepsilon_{t+1}, \end{aligned}$$

- Solve for

$$x_{t+1} = Px_t + Qz_t$$

  and

$$y_t = Rx_t + Sz_t.$$

Seigniorage: results

$\bar{g}$	0	.01	.1
P	[.9697]	[.9697]	[.9697]
Q	$\begin{bmatrix} .07580 & 0 \end{bmatrix}$	$\begin{bmatrix} .07580 & 0 \end{bmatrix}$	$\begin{bmatrix} .07580 & 0 \end{bmatrix}$
R	$\begin{bmatrix} -0.4300 \\ 0.4781 \\ -0.4782 \\ 0 \\ -0.3282 \end{bmatrix}$	$\begin{bmatrix} -0.4300 \\ 0.4781 \\ -0.4782 \\ -0.0053 \\ -0.3282 \end{bmatrix}$	$\begin{bmatrix} -0.4300 \\ 0.4781 \\ -0.4782 \\ -0.0591 \\ -0.3282 \end{bmatrix}$
S	$\begin{bmatrix} 0.2536 & 0 \\ 0.5802 & 0 \\ -0.5802 & 0 \\ 0 & 0 \\ 1.1662 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.2536 & 0 \\ 0.5802 & 0 \\ -0.5802 & 0 \\ -0.0065 & 0.0111 \\ 1.1662 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.2536 & 0 \\ 0.5802 & 0 \\ -0.5802 & 0 \\ -0.0717 & 0.1235 \\ 1.1662 & 0 \end{bmatrix}$