1 Solving matrix quadratic equations

Solving matrix quadratic equations

- We look for a solution to the quadratic equation,

$$AP^2 - BP - C = 0$$

- of the form $P = \Psi\Lambda\Psi^{-1}$
- where Λ is a matrix of eigenvalues on the diagonal of the form

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \cdots & \ddots & \lambda_{n-1} & 0 \\ 0 & \cdots & \cdots & 0 & \lambda_n \end{bmatrix},$$

- Ψ is a matrix with the corresponding eigenvectors.
- This way of writing P gives $P^2 = \Psi\Lambda\Psi^{-1}\Psi\Lambda\Psi^{-1} = \Psi\Lambda^2\Psi^{-1}$

Solving matrix quadratic equations

- The matrices A, B, and C of $AP^2 - BP - C = 0$ are all $n \times n$.
- Construct the $2n \times 2n$ matrices

$$D = \begin{bmatrix} B & C \\ I & \vec{0} \end{bmatrix}$$

and

$$E = \begin{bmatrix} A & \vec{0} \\ \vec{0} & I \end{bmatrix}$$

Solving matrix quadratic equations

- Find the solution to the generalized eigenvalue problem for the matrix pair (D, E).
- The solution to this problem is a set of $2n$ eigenvalues λ_k and corresponding eigenvectors x_k, such that

$$Dx_k = Ex_k\lambda_k$$

- Assume that there are at least n stable eigenvectors, those whose absolute value is less than one.

- Order the eigenvalues and their corresponding eigenvectors, so that the n stable eigenvalues come first.

Solving matrix quadratic equations

- The eigenvectors are columns, so that the matrix X is

$$X = \left[\begin{array}{ccccc} x_{1,1} & x_{2,1} & \cdots & \cdots & x_{2n,1} \\ x_{1,2} & x_{2,2} & \vdots & \vdots & x_{2n,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{1,2n} & x_{2,2n} & \cdots & \cdots & x_{2n,2n} \end{array} \right].$$

Solving matrix quadratic equations
Partitian X so that

$$X = \left[\begin{array}{cc} X^{11} & X^{21} \\ X^{12} & X^{22} \end{array} \right] =$$

$$\left[\begin{array}{cc} \left[\begin{array}{cccc} x_{1,1} & x_{2,1} & \cdots & x_{n,1} \\ x_{1,2} & x_{2,2} & \cdots & x_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1,n} & x_{2,n} & \cdots & x_{n,n} \end{array} \right] & \left[\begin{array}{cccc} x_{n+1,1} & x_{n+2,1} & \cdots & x_{2n,1} \\ x_{n+1,2} & x_{n+2,2} & \cdots & x_{2n,2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n+1,n} & x_{n+2,n} & \cdots & x_{2n,n} \end{array} \right] \\ \left[\begin{array}{cccc} x_{1,n+1} & \cdots & \cdots & x_{n,n+1} \\ x_{1,n+2} & \ddots & \cdots & x_{n,n+2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1,2n} & \cdots & \cdots & x_{n,2n} \end{array} \right] & \left[\begin{array}{cccc} x_{n+1,n+1} & \cdots & \cdots & x_{2n,n+1} \\ x_{n+1,n+2} & \ddots & \cdots & x_{2n,n+2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n+1,2n} & \cdots & \cdots & x_{2n,2n} \end{array} \right] \end{array} \right]$$

.

- The generalized eigenvalues gives the problem in the form

$$\left[\begin{array}{cc} B & C \\ I & \overrightarrow{0} \end{array} \right] \left[\begin{array}{cc} X^{11} & X^{21} \\ X^{12} & X^{22} \end{array} \right] = \left[\begin{array}{cc} A & \overrightarrow{0} \\ \overrightarrow{0} & I \end{array} \right] \left[\begin{array}{cc} X^{11} & X^{21} \\ X^{12} & X^{22} \end{array} \right] \left[\begin{array}{cc} \Delta^1 & \overrightarrow{0} \\ \overrightarrow{0} & \Delta^2 \end{array} \right]$$

- Multiplying out the matrices on each side gives

$$\left[\begin{array}{cc} BX^{11} + CX^{12} & BX^{21} + CX^{22} \\ X^{11} & X^{21} \end{array} \right] = \left[\begin{array}{cc} AX^{11}\Delta^1 & AX^{21}\Delta^2 \\ X^{12}\Delta^1 & X^{22}\Delta^2 \end{array} \right]$$

Solving matrix quadratic equations

- Looking at corresponding partitions, we use

$$X^{11} = X^{12}\Delta^1,$$

and

$$BX^{11} + CX^{12} = AX^{11}\Delta^1.$$

- Substituting in $X^{12}\Delta^1$ for X^{11} in the second equation gives

$$BX^{12}\Delta^1 + CX^{12} = AX^{12}\Delta^1\Delta^1,$$

- postmultiplying both sides by $\left(X^{12}\right)^{-1}$ gives

$$BX^{12}\Delta^1\left(X^{12}\right)^{-1} + C = AX^{12}\Delta^1\Delta^1\left(X^{12}\right)^{-1}.$$

- Define $P = X^{12}\Delta^1\left(X^{12}\right)^{-1}$.
- Then $P^2 = X^{12}\Delta^1\Delta^1\left(X^{12}\right)^{-1}$ and, from above,

$$BP + C = AP^2.$$

Solving matrix quadratic equations

Therefore, the solution to the matrix quadratic equation can be found by constructing the matrices D and E and finding the solution to the generalized eigenvalue problem for those matrices as the generalized eigenvector matrix X and the generalized eigenvalue matrix Δ (ordered appropriately, with the stable eigenvalues first). The matrix Δ^1, contains the eigenvalues and the matrix X^{12} contains the eigenvectors that we use to construct

$$P = X^{12}\Delta^1\left(X^{12}\right)^{-1}.$$

Solving matrix quadratic equations

- Summarized
  - To solve: form matrices D and E
  - Use Matlab program *eig* in the form

  $[V, D] = eig(A, B)$ produces a diagonal matrix D of generalized eigenvalues and a full matrix V whose columns are the corresponding eigenvectors so that $A * V = B * V * D$

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  - Select the eigenvalues with absolute values less than one
  - Select the corresponding eigenvectors
  - Use these to make the matrix X^{11}, X^{12}, and Δ^1