A model of learning with idiosyncratic measurement error aka: The importance of good public information

Prof. McCandless
UCEMA

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1 Lecture on Learning paper

.Idea

- Start with a very simple model (hansen's rbc model)
- have households learn about aggregate values of variables with measurement noise
- this measurement noise is idiosyncratic
- Each household estmates forecasting equations from noisy data
- Forecasting parameters are biased
- Find Stationary state with idiosyncratic noise
- find dynamic model with idiosyncratic noise

.The model (Hansen)

- Households max

$$E_t^j \sum_{i=0}^{\infty} \beta^i \left(\ln c_{t+i}^j + A \ln \left(1 - h_{t+i}^j \right) \right)$$

  subject to

$$k_{t+1}^j = w_t h_t^j + (r_t + 1 - \delta) k_t^j - c_t^j + \chi_t^j,$$

- Production is competitive and the production function is Cobb-Douglas

$$Y_t = \lambda_t K_t^{\theta} H_t^{1-\theta}$$

- factor markets result in

$$r_t = \theta \frac{Y_t}{K_t}$$

.The full model

- The Hansen RBC model can be written in terms of the following five equations :

$$\begin{aligned}
\frac{1}{\beta c_t^j} &= E_t \frac{r_{t+1}^j + 1 - \delta}{c_{t+1}^j}, \\
(1-\theta)\frac{Y_t}{H_t} &= \frac{AC_t}{1-H_t}, \\
K_{t+1} + C_t &= Y_t + (1-\delta)K_t, \\
Y_t &= \lambda_t K_t^{\theta} H_t^{1-\theta}, \\
r_t &= \theta \frac{Y_t}{K_t}.
\end{aligned}$$

.Forecasting equations (1)

- Households estimate the forecasting equations

$$c_{t+1}^j = \varphi_{11} k_{t+1}^j + \varphi_{12} y_t^j$$

and

$$r_{t+1}^j = \varphi_{21} k_{t+1}^j + \varphi_{22} y_t^j$$

- estimated with the noisy data that they have
- Household j has the data history comprised of

$$k_{t+1}^j = K_{t+1} + \varepsilon_t^{j,k}$$

and

$$y_t^j = Y_t + \varepsilon_t^{j,y}.$$

.Forecasting equations (2)

- Define

$$X = \left[\begin{array}{cc} k_s^j & y_{s-1}^j \end{array}\right] = \left[\begin{array}{cc} \left(K_s + \varepsilon_s^{j,k}\right) & \left(Y_{s-1} + \varepsilon_{s-1}^{j,y}\right) \end{array}\right]$$

- and

$$Y = \left[\begin{array}{cc} c_s^j & r_s^j \end{array}\right] = \left[\begin{array}{cc} \left(C_s + \varepsilon_s^{j,c}\right) & \left(r_s + \varepsilon_s^{j,r}\right) \end{array}\right].$$

- The parameters for the forecasting model are found from

$$\Phi = \left[\begin{array}{cc} \varphi_{11} & \varphi_{21} \\ \varphi_{12} & \varphi_{22} \end{array}\right]$$

- are found from

$$\Phi = \left(X'X\right)^{-1} X'Y.$$

Forecasting equations (3)

- This equation can be written as

$$\begin{aligned}
& \left(X'X\right)^{-1} X'Y \\
= & \left(\left[\begin{array}{c} \left(K_s + \varepsilon_s^{j,k}\right) \\ \left(Y_{s-1} + \varepsilon_{s-1}^{j,y}\right) \end{array} \right] \left[\begin{array}{cc} \left(K_s + \varepsilon_s^{j,k}\right) & \left(Y_{s-1} + \varepsilon_{s-1}^{j,y}\right) \end{array} \right] \right)^{-1} \\
& \times \left[\begin{array}{c} \left(K_s + \varepsilon_s^{j,k}\right) \\ \left(Y_{s-1} + \varepsilon_{s-1}^{j,y}\right) \end{array} \right] \left[\begin{array}{cc} \left(C_s + \varepsilon_s^{j,c}\right) & \left(r_s + \varepsilon_s^{j,r}\right) \end{array} \right],
\end{aligned}$$

- Because the shocks are independent, this becomes

$$\begin{aligned}
\Phi &= \left(X'X\right)^{-1} X'Y \\
&= \left[\begin{array}{cc} K_s^2 + \sigma_{\varepsilon_k}^2 & K_s Y_{s-1} \\ K_s Y_{s-1} & Y_{s-1}^2 + \sigma_{\varepsilon_y}^2 \end{array} \right]^{-1} \left[\begin{array}{cc} K_s C_s & K_s r_s \\ Y_{s-1} C_s & Y_{s-1} r_s \end{array} \right].
\end{aligned}$$

Forecasting equations (4)

- Since the $X'X$ matrix is only 2×2, it can be inverted exactly and after a bit of substitution, along with the assumption that the variances are proportionally the same for all variables, that $\sigma_{\varepsilon_k}^2 = K^2 \sigma_\varepsilon^2$ and $\sigma_{\varepsilon_y}^2 = Y^2 \sigma_\varepsilon^2$, one gets stationary state parameters for the forecasting equations of

$$\Phi = \left[\begin{array}{cc} \frac{\overline{C}}{(2+\sigma_\varepsilon^2)\overline{K}} & \frac{\overline{r}}{(2+\sigma_\varepsilon^2)\overline{K}} \\ \frac{\overline{C}}{(2+\sigma_\varepsilon^2)\overline{Y}} & \frac{\overline{r}}{(2+\sigma_\varepsilon^2)\overline{Y}} \end{array} \right].$$

Applying the forecasting equations to the model (1)

- The first equation of the model is

$$\frac{1}{\beta c_t^j} = E_t \frac{r_{t+1}^j + 1 - \delta}{c_{t+1}^j}$$

- Apply the equations for forecasting to get

$$\begin{aligned}
\frac{1}{\beta c_t^j} &= E_t \frac{r_{t+1}^j + 1 - \delta}{c_{t+1}^j}, \\
&= \frac{\varphi_{21} k_{t+1}^j + \varphi_{22} y_t^j + 1 - \delta}{\varphi_{11} k_{t+1}^j + \varphi_{12} y_t^j} \\
\frac{1}{\beta c_t^j} &= \frac{\varphi_{21} \left(K_{t+1} + \varepsilon_t^{j,k}\right) + \varphi_{22} \left(Y_t + \varepsilon_t^{j,y}\right) + 1 - \delta}{\varphi_{11} \left(K_{t+1} + \varepsilon_t^{j,k}\right) + \varphi_{12} \left(Y_t + \varepsilon_t^{j,y}\right)}.
\end{aligned}$$

Applying the forecasting equations to the model (2)

- Both sides of the last equation can be inverted and we get

$$\beta c_t^j = \frac{\varphi_{11}\left(K_{t+1}+\varepsilon_t^{j,k}\right)+\varphi_{12}\left(Y_t+\varepsilon_t^{j,y}\right)}{\varphi_{21}\left(K_{t+1}+\varepsilon_t^{j,k}\right)+\varphi_{22}\left(Y_t+\varepsilon_t^{j,y}\right)+1-\delta}.$$

- aggregating the lhs is easy and gives βC_t
- Aggregating the rhs is quite difficult and is done by approximation

Applying the forecasting equations to the model (3)

- Taking a second order Taylor approximation of the rhs gives

$$\begin{aligned}\beta C_t &= \frac{EC_{t+1}}{Er_{t+1}+1-\delta} \\ &\quad +\varphi_{21}\frac{\varphi_{21}EC_{t+1}-\varphi_{11}\left(Er_{t+1}+1-\delta\right)}{\left(Er_{t+1}+1-\delta\right)^3}\sigma^2_{\varepsilon_k} \\ &\quad +\varphi_{22}\frac{\varphi_{22}EC_{t+1}-\varphi_{12}\left(Er_{t+1}+1-\delta\right)}{\left(Er_{t+1}+1-\delta\right)^3}\sigma^2_{\varepsilon_y}.\end{aligned}$$

replacing the parameters of the forecasting equations with what they equal gives

$$\beta\overline{C} = \frac{\frac{2}{2+\sigma_\varepsilon^2}\overline{C}}{\frac{2}{2+\sigma_\varepsilon^2}\overline{r}+1-\delta} - 2\frac{\frac{\overline{C}\overline{r}}{(2+\sigma_\varepsilon^2)^2}\left(1-\delta\right)}{\left(\frac{2}{2+\sigma_\varepsilon^2}\overline{r}+1-\delta\right)^3}\sigma_\varepsilon^2$$

Applying the forecasting equations to the model (4)

- Cancelling out $\overline{C}$, gives

$$\beta = \frac{\frac{2}{2+\sigma_\varepsilon^2}}{\frac{2}{2+\sigma_\varepsilon^2}\overline{r}+1-\delta} - 2\frac{\frac{\overline{r}}{(2+\sigma_\varepsilon^2)^2}\left(1-\delta\right)}{\left(\frac{2}{2+\sigma_\varepsilon^2}\overline{r}+1-\delta\right)^3}\sigma_\varepsilon^2$$

- Notice that if $\sigma_\varepsilon^2 = 0$, this equation becomes the standard expression for this first order condition.

The stationary state (1)
The five equations for the stationary state are

$$
\begin{aligned}
\beta &= \frac{\frac{2}{2+\sigma_\varepsilon^2}}{\frac{2}{2+\sigma_\varepsilon^2}\overline{r}+1-\delta} - 2\frac{\frac{\overline{r}}{(2+\sigma_\varepsilon^2)^2}(1-\delta)}{\left(\frac{2}{2+\sigma_\varepsilon^2}\overline{r}+1-\delta\right)^3}\sigma_\varepsilon^2 \\
(1-\theta)\frac{\overline{Y}}{\overline{H}} &= \frac{A\overline{C}}{1-\overline{H}}, \\
\overline{C} &= \overline{Y}-\delta\overline{K}, \\
\overline{Y} &= \overline{K}^{\theta}\overline{H}^{1-\theta}, \\
\overline{r} &= \theta\frac{\overline{Y}}{\overline{K}}.
\end{aligned}
$$

.The stationary state (2)

- For the parameter values of $\beta = .99$, $\delta = .025$, $\theta = .36$, and $A = 1.72$
- The stationary state values when $\sigma_\varepsilon^2 = 0$ are the usual

$\overline{K} = 12.6695$,
$\overline{Y} = 1.2353$,
$\overline{C} = .9186$,
$\overline{H} = .3335$,
$\overline{r} = .0351$.

.The stationary states with idiosyncratic measurement error

- The effects of bias in the forecasting estimations

.The stationary states with idiosyncratic measurement error

- The effects of bias in the forecasting estimations

.The dynamic version of the model

- The log-linear version of the basic model is

$$
\begin{aligned}
0 &= \widetilde{c}_t^j - E_t\widetilde{c}_{t+1}^j + \beta\overline{r}E_t\widetilde{r}_{t+1}^j, \\
0 &= \widetilde{Y}_t - \frac{\widetilde{h}_t^j}{1-\overline{H}} - \widetilde{c}_t^j, \\
0 &= \overline{Y}\widetilde{Y}_t - \overline{C}\widetilde{C}_t + \overline{K}\left[(1-\delta)\widetilde{K}_t - \widetilde{K}_{t+1}\right], \\
0 &= \widetilde{\lambda}_t + \theta\widetilde{K}_t + (1+\theta)\widetilde{H}_t - \widetilde{Y}_t, \\
0 &= \widetilde{Y}_t - \widetilde{K}_t - \widetilde{r}_t,
\end{aligned}
$$

- and

$$
\widetilde{\lambda}_t = \gamma\widetilde{\lambda}_{t-1} + \widetilde{\varepsilon}_t^{\lambda}.
$$

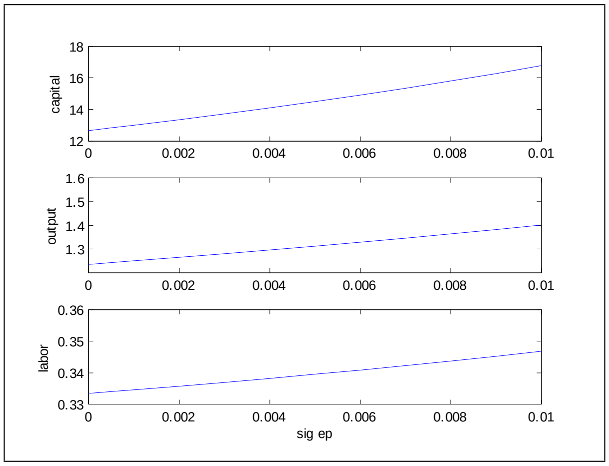

Figure 1: Stationary state values for $\overline{K}$, $\overline{Y}$, and $\overline{H}$ as a function of the measurement error

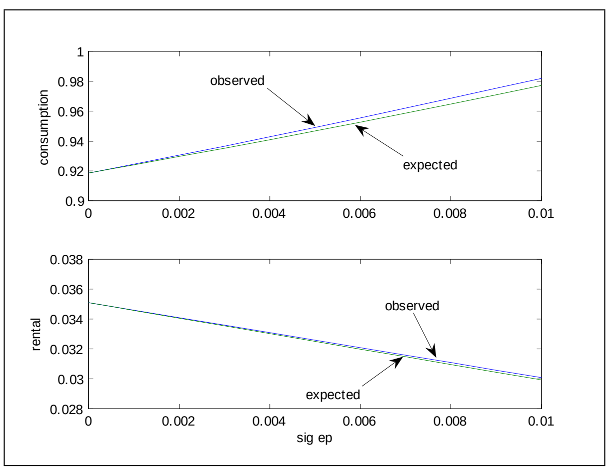

Figure 2: Stationary state values for $\overline{C}$, $\overline{r}$, EC_{t+1}, and Er_{t+1} as function of measurement error

.The dynamic version of the model

- The expectational variables, $E_t\widetilde{c}^j_{t+1}$ and $E_t\widetilde{r}^j_{t+1}$ are determined by each family using an OLS model and using $\widetilde{K}_{t+1}$ and $\widetilde{Y}_t$ as explanatory variables
- the equation is

$$\left[\begin{array}{cc} E_t\widetilde{C}_{t+1} & E_t\widetilde{r}_{t+1}\end{array}\right] = \left[\begin{array}{cc} \varphi(t-1)_{11} & \varphi(t-1)_{21} \\ \varphi(t-1)_{12} & \varphi(t-1)_{22}\end{array}\right]\left[\begin{array}{cc} \widetilde{K}^j_{t+1} & \widetilde{Y}^j_t\end{array}\right],$$

- where the $\varphi(t-1)_{11}$ are estimated using data available up to time t-1
- The OLS estimation is

$$\Phi = \left(\left(X^j\right)' X^j\right)^{-1}\left(X^j\right)' Y^j$$

.The dynamic version of the model

- Given the measurement error, a bias is introduced into the estimates
- The bias can be seen in

$$\begin{aligned}\Phi &= \left(\left[\begin{array}{cc} var\widetilde{K} & cov\widetilde{K}\widetilde{Y} \\ cov\widetilde{K}\widetilde{Y} & var\widetilde{Y}\end{array}\right] + \left[\begin{array}{cc} \sigma^2_{\mu^K} & 0 \\ 0 & \sigma^2_{\mu^Y}\end{array}\right]\right)^{-1} \\ & \times \left[\begin{array}{cc} cov\widetilde{K}\widetilde{C} & cov\widetilde{K}\widetilde{r} \\ cov\widetilde{Y}\widetilde{C} & cov\widetilde{Y}\widetilde{r}\end{array}\right].\end{aligned}$$

.The dynamic version of the model (solving the model)

- A state space version of the model is used
- the variables are

$$x_t = \left[\begin{array}{cccccccc} \widetilde{K}_{t+1} & \widetilde{H}_t & \widetilde{Y}_t & \widetilde{C}_t & \widetilde{r}_t & E_t\widetilde{C}_{t+1} & E_t\widetilde{r}_{t+1} & \widetilde{\lambda}_t\end{array}\right]$$

- and the state space formulation of the problem is

$$A_t(\Phi_{t-1})\,x_t = B_t(\Phi_{t-1})\,x_{t-1} + C\varepsilon_t,$$

- the recursive updating equation for the parameters is

$$\left[\begin{array}{c} \Phi_t \\ P_t\end{array}\right] = G\left(\left[\begin{array}{c} \Phi_{t-1} \\ P_{t-1}\end{array}\right], x_t\right).$$

.The dynamic version of the model (solving the model)

- the state space model can be solved directly in each period as

$$x_t = [A_t(\Phi_{t-1})]^{-1} B_t(\Phi_{t-1}) x_{t-1} + [A_t(\Phi_{t-1})]^{-1} C\varepsilon_t.$$

- where

$$A_t = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & -1 & \beta\overline{r} & 0 \\ 0 & -\frac{1}{1-\overline{H}} & 1 & -1 & 0 & 0 & 0 & 0 \\ -\overline{K} & 0 & \overline{Y} & -\overline{C} & 0 & 0 & 0 & 0 \\ 0 & 1+\theta & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ -\varphi_{11}^1(t-1) & 0 & -\varphi_{12}^1(t-1) & 0 & 0 & 1 & 0 & 0 \\ -\varphi_{21}^1(t-1) & 0 & -\varphi_{22}^1(t-1) & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

.The dynamic version of the model (solving the model)

$$B_t = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -(1-\delta)\overline{K} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\theta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma \end{bmatrix},$$

and

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}'.$$

.Finding an impulse response function

- Constant gain OLS can be very slow to converge
- the economy is run for 40,000 periods with a forgetting factor of .99999
- then run again using the average values for Φ over the last 20,000 periods as a new starting point.
- The economy is run twice for 40,199 periods beginning with the coefficients found above (with the forgetting factor = 1).
- the same normally distributed shocks are applied to the economy
- except that in period 40,001 of the second running, an additional impulse of .1 is applied to the technology shock
- The impulse response function to a technology shock for this economy is found by subtracting the last 199 observations of the first run without forgetting from the second run.

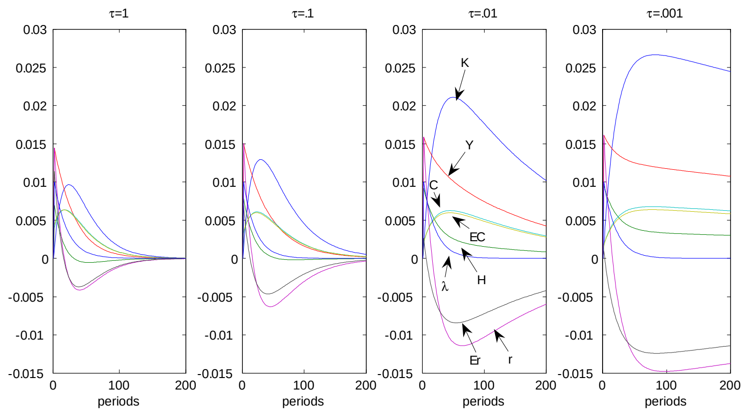

Figure 3: Impulse-response functions

.How measurement error affects the parameters

- For an example economy with a variance of measurement error of .001,
- The range of the signal to noise ratio is from 10 to .0001

τ	10	1	.1	.01	.001	.0001
φ_{11}	0.4989	0.4856	0.4295	0.3258	0.2260	0.1978
φ_{12}	0.2892	0.2763	0.2698	0.2053	0.1163	0.0914
φ_{21}	-1.0079	-0.9936	-0.8935	-0.6341	-0.4987	-0.5149
φ_{22}	0.9702	0.9583	0.8572	0.5134	0.2101	0.1088

.Impulse response functions

- τ is the ratio of $\sigma^2_{\varepsilon^\lambda}$ to σ^2_μ, shock is to technology

.Value of public information

- Reliable public information can permit returning to the rational expectations equilibrium
- Only information necessary is survey data from households
  - that averages the data observed by the households
- Why

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