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ARBITRAGING MISPRICED ASSETS WITH SEPARATION PORTFOLIOS TO LESSEN TOTAL RISK

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Abstract

This paper expands on a procedure to arbitrage mispriced assets against the benchmark provided by the Security Market Line, but using only separation portfolios to put up a feasible portfolio with the same beta as the mispriced asset and the least total risk among other alternative portfolios. Coming next, such arbitrage is dealt directly with one single separation portfolio, which grants that the total risk linked with the arbitrage portfolio equals the non-systematic risk conveyed by the mispriced asset.

JEL: G10, G11, G13

Key Words: arbitrage portfolios, separation portfolios, total risk, systematic risk

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INTRODUCTION

The arbitrage of mispriced assets against portfolios lying in the Security Market Line (SML) has become one of the most cherished applications of that model, bringing into light a clear-cut example of what is meant by arbitraging against a benchmark. A well-balanced development of this approach can be followed in Elton-Gruber (1995,1997), whereas a wide background on arbitrage portfolios can be found in Apreda (2001).

Arbitraging against the SML shows a key feature: we pick a portfolio belonging to it, carrying the same beta as the mispriced asset. Following this path, the paper expands on how arbitrage portfolios can be put up by joining the mispriced asset with a separation portfolio that performs as the counterpart lying in the SML. Separation portfolios are also priced in the SML, although they have their natural address in the Capital Market Line (CML). This procedure exhibits two useful characteristics:

a) The counterpart is simply framed by means of the risk-free asset and the portfolio market.

b) Although arbitrage portfolios should be risk-free when the mispriced asset and the counterpart are measured out with beta, it is total risk that could bring a wedge in real arbitrage opportunities, hindering any likely hedging strategy grounded on beta values only. The procedure grants that such counterpart bears less total risk than the mispriced asset.

In section 1 we are going to briefly expand on the conventional setting in which arbitrage portfolios can be depicted, giving heed to some of its customary shortcomings. Section 2 brings home the way arbitrage can proceed by taking two separation portfolios to set up a counterpart to the mispriced asset in the SML. Such counterpart exhibits less total risk than any other likely portfolio or asset with the same beta than the mispriced asset. Lastly, in section 3, we prove that we only need one separation portfolio to fashion an arbitrage portfolio whose total risk equals the non systematic risk of the mispriced asset.

1. THE CONVENTIONAL SETTING

Let us suppose that we find out that a financial asset **A** is undervalued with regard to the SML. This means that, given the systematic level of risk conveyed by **A,** it holds

$$\mathbf{E[\,R(A)\,] \;>\; R(F) \;+\; <E[\,R(M)\,] - R(F)> .\ \beta_A}$$

Therefore, an arbitrage portfolio $\mathbf{\Delta P}$ can be devised along the following lines:

i) We single out two accessible assets (or portfolios) **B** and **C**, both of them lying in the **SML**, so as to get a portfolio with the following features:

$$\begin{cases} \mathbf{D} = \langle x_B ; x_C \rangle \quad ; \quad x_B + x_C = 1 \\ \beta_D = \beta_A \\ E[R(D)] = R(F) + \langle E[R(M)] - R(F) \rangle . \beta_A \end{cases}$$

ii) Once this portfolio **D** is figured out, we proceed to design a new portfolio **ΔP** whose structure is:

$$\Delta P = \langle x_A ; x_D \rangle = \langle +1 ; -1 \rangle$$

that is to say, this portfolio claims a long position in asset **A** and a short position, for the same level of wealth, in portfolio **D**. Therefore, it is a self-financing portfolio whose beta amounts to zero since

$$\beta_{\Delta P} = x_A \beta_A + x_D \beta_D = 1 \beta_A + (-1) \beta_A = 0$$

iii) Last of all, as **A** is undervalued in terms of the **SML**, a positive differential expected rate of return for **ΔP** is granted :

$$E[R(\Delta P)] = E[R(A)] - E[R(D)] > 0$$

Although from a theoretical point of view this is a flawless procedure, there are some shortcomings as soon as we try to make it available in practice. We would like to highlight some of them:

a) **B** or **C** could not be ready for use in the market at date "**t**".

b) Regulations may prevent economic agents from purchasing some assets (this could be an insurmountable problem for pension funds if they run the risk of overinvesting in **B** or **C**).

c) Transaction costs in a broad sense (trading, information, financial, microstructure costs, and taxes) could be greater that the expected differential rate the arbitrage portfolio is bound to get (Apreda, 2000).

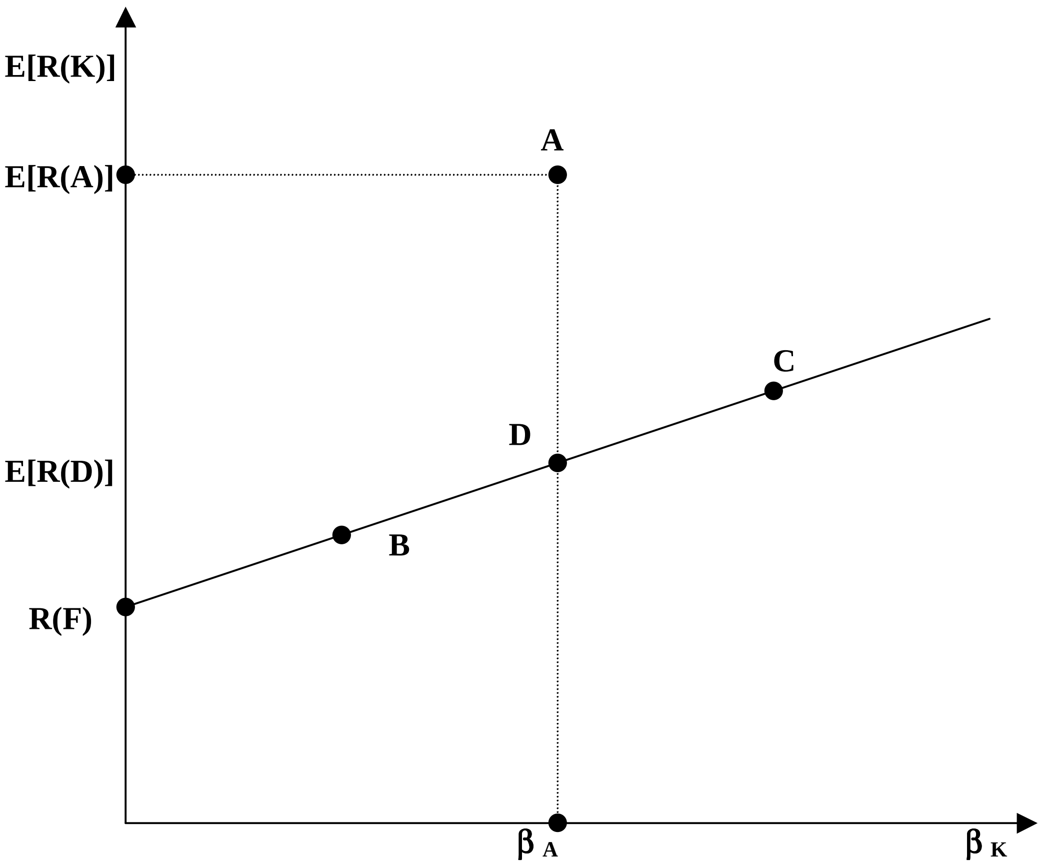
d) In case that β_A be higher or lower than both β_B and β_C, then either of the participations x_B or x_C become negative. But shortselling could be prevented or banished outright. Even if it were feasible, related transaction costs would rule it out eventually.

e) Whereas **D** lies in the SML where the distinctive risk is systematic only, **D** could actually bear more total risk than **A**, turning the expected return of **ΔP** riskier eventually (more on this in section 2).

Remark:

Although along this paper we are going to highlight the mispriced asset as one being undervalued, expansion on the mispriced asset when it is overvalued follows along the same lines.

Picture 1: The conventional setting for an arbitrage portfolio



Remark: ***Asset A is mispriced, with an expected return higher than what might be expected in equilibria, becoming thus cheaper than portfolio D.***

$$E[\,R(D)\,] = R(F) + < E[\,R(M)\,] - R(F) > .\ \beta_A < E[\,R(A)\,]$$

2. THE CML AS A BRIDGE TO ARBITRAGE PORTFOLIOS

Instead of picking up two assets **B** and **C** lying in the SML at the addresses

$$\mathbf{B = (\beta_B \,;\, E[R(B)])} \qquad \text{and} \qquad \mathbf{C = (\beta_C \,;\, E[R(C)])}$$

as it is suggested by the conventional setting, let us stick to the following procedure instead

i) Choose two separation portfolios SB and SC

ii) The addresses of SB and SC are, respectively, the ones of B and C. That is to say:

$$\mathbf{SB = (\beta_B \,;\, E[R(B)])} \qquad \text{and} \qquad \mathbf{SC = (\beta_C \,;\, E[R(C)])}$$

Although these portfolios are located at the same place as **B** and **C** (see picture 2), they also belong to the Capital Market Line (CML). Therefore, we could define a portfolio **D**

(1) $$\mathbf{D = < x_{SB} \,;\, x_{SC} > \quad ; \quad x_{SB} + x_{SC} = 1}$$

subject to the constraints:

$$\beta_D = \beta_A$$

and

$$\mathbf{E[\,R(D)\,] = R(F) + < E[\,R(M)\,] - R(F) > . \,\beta_A}$$

Let us focus on the CML for a while. In such environment we know that the structure of portfolios **SB** and **SC** consists of a fraction of the initial wealth in the market portfolio **M** and the remainder in the risk-free asset **F**. Hence, it holds:

$$\mathbf{SB = < x_{FB} \,;\, x_{MB} > \quad ; \quad x_{FB} + x_{MB} = 1}$$

and

$$\mathbf{SC = < x_{FC} \,;\, x_{MC} > \quad ; \quad x_{FC} + x_{MC} = 1}$$

Firstly, there is a simple relationship between the total risk measured in the world of the CML and the fractional part to be held in the market portfolio:

(2) $$\sigma_{SB} = x_{MB} \,.\, \sigma_M$$

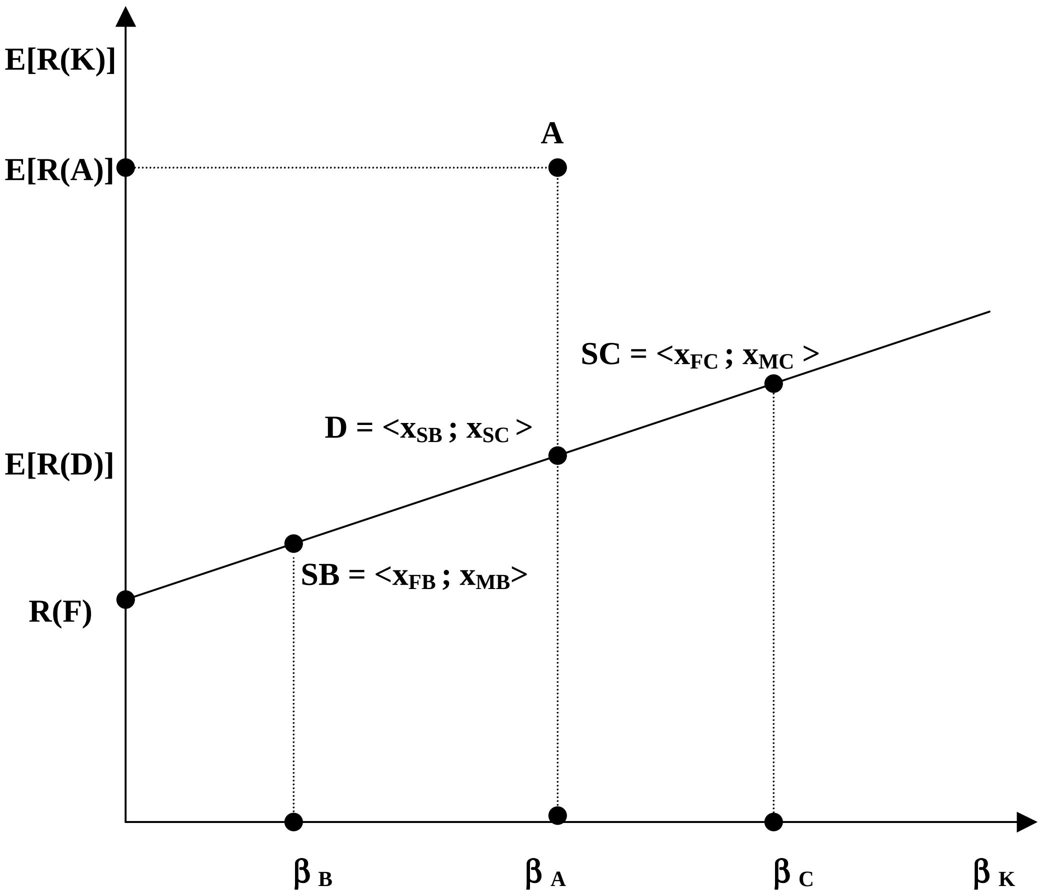
(3) $$\sigma_{SC} = x_{MC} \,.\, \sigma_M$$

secondly, it follows from these expressions (see the Appendix) that the proportions in the market portfolio are even to betas

(4) $$x_{MB} = \beta_{SB}$$

(5) $$x_{MC} = \beta_{SC}$$

Picture 2: Arbitraging with separation portfolios



Remark: ***D is a portfolio lying in the SML, set up as a mix of two separation portfolios SB and SC with natural address at the CML, but also priced by the SML.***

The lemma that comes next proves that such portfolio **D** is workable and fits together with the mispriced asset to round off an arbitrage portfolio.

Lemma 1: For any mispriced asset A we can devise a portfolio D as defined in (1). Besides, both A and D put together an arbitrage portfolio.

Proof: As **D** lies in the SML, it holds that

$$\beta_D = x_{SB} \cdot \beta_{SB} + x_{SC} \cdot \beta_{SC}$$

Plugging (4) and (5) in the former equation, we get

$$\beta_D = x_{SB} \cdot x_{MB} + x_{SC} \cdot x_{MC}$$

but

$$\beta_D = \beta_A$$

Now, we have to determine the fractions of the investor's funds allocated for **SB** and **SC**. Recalling that

$$x_{SC} = 1 - x_{SB}$$

we get

$$\beta_D = \beta_A = x_{SB} \cdot x_{MB} + (1 - x_{SB}) \cdot x_{MC}$$

Calculations for x_{SB} and x_{SC} result in

$$x_{SB} = \{ \beta_A - x_{MC} \} / \{ x_{MB} - x_{MC} \}$$

and

$$x_{SC} = \{ x_{MB} - \beta_A \} / \{ x_{MB} - x_{MC} \}$$

It is clear that **D** and **A** put together an arbitrage portfolio, by construction. ¶¶

It seems worthy of being remarked that portfolios **SB** and **SC** have a desirable feature: among all portfolios in the SML which share their expected return and beta: they are the ones with the least total risk, as the following lemma makes clear.

Lemma 2: Among all portfolios lying in the SML at the address

$$(\beta_B ; E[R(B)])$$

SB is the one with the least total risk.

Proof: The total risk of SB conveys systematic and non-sistematic risk components:

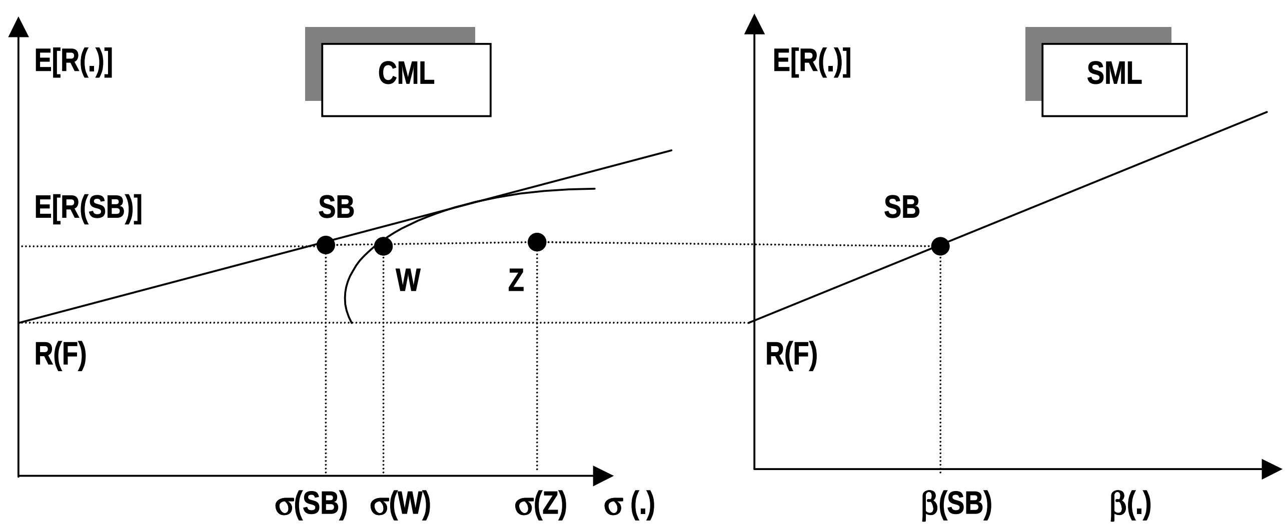
(6) $$\sigma^2_{SB} = \beta^2_{SB} \cdot \sigma^2_M + \sigma^2(\varepsilon)_{SB}$$

where $\sigma^2(\varepsilon)_{SB}$ stands for the residual variance of portfolio **SB**. This equation breaks the total risk into two components, that part due to systematic risk (the first term in the right hand side), and that part due to non-systematic risk (the second term). It can be found in the Appendix that the variance of **SB** can be written as

(7) $$\sigma^2_{SB} = \beta^2_{SB} \cdot \sigma^2_M$$

Contrasting (6) with (7), it follows that the residual variance is zero.

Picture 3: Relationship between the CML and the SML



Remark: ***SB at SML is a cluster of several portfolios and assets with the same expected return and different total risks. In other words, they convey the same systematic risk but not the same total risk, due to non-systematic risk.***

On the other hand (see picture 3), for any other portfolio **Z** (even the efficient portfolio **W**) with the same expected return and beta, the residual variance is not zero, because

$$\sigma^2_Z = \beta^2_{SB} \cdot \sigma^2_M + \sigma^2(\varepsilon)_Z > \sigma^2_{SB}$$

Therefore, **SB** is the portfolio with the least total risk. ¶¶

By the same token, it can be proved that **SC** is the portfolio with the least total risk for all portfolios lying in the SML at the address

$$(\beta_C ; E[R(C)])$$

3. ARBITRAGE AGAINST A SINGLE SEPARATION PORTFOLIO

There is still another way to build up portfolio **D** to arbitrage against asset **A**, and much simpler than the method provided in the foregoing section.

When facing an environment in which asset **A** is undervalued (picture 1), we proceed to design a separation portfolio **S** as follows:

(9) $$S = < x_F ; x_M > \quad ; \quad x_F + x_M = 1$$

subject to the restrictions

$$\beta_S = \beta_A$$

and

$$E[R(S)] = R(F) + < E[R(M)] - R(F) > . \beta_A$$

This portfolio actually exists and allows for the raising of an arbitrage portfolio is the subject of the next lemma.

Lemma 3: The portfolio S defined in (8) is feasible and we can set up an arbitrage portfolio with both S and the mispriced asset A.

Proof: In the CML, it holds that

$$\sigma_S = x_M \cdot \sigma_M$$

and its expected returns translated by the CML yields

$$E[R(S)] = R(F) + < \{ E[R(M)] - R(F) \} / \sigma^2_M > . \sigma_S$$

by plugging (2) in this expression we obtain

$$\mathbf{E[\ R(S)\ ]\ =\ R(F)\ +\ <\{\ E[\ R(M)\ ]\ -R(F)\ \}\ /\ \sigma^2_M\ >\ .\ x_M\ .\ \sigma_M}$$

and $\mathbf{x_M}$ is solvable, namely,

$$\mathbf{x_M\ =\ <\ E[\ R(S)\ ]-R(F)\ >\ /\ <\{\ E[\ R(M)\ ]\ -\ R(F)\ \}\ /\ \sigma_M\ >}$$

So, by taking

$$\mathbf{x_F\ =\ 1\ -x_M}$$

S is endowed with the following structure

$$\mathbf{S\ =\ <\ x_F\ ;\ x_M\ >}$$

To show how **S** and **A** bring about an arbitrage portfolio, let **ΔP** be defined as

$$\mathbf{\Delta P\ =\ <\ x_A\ ;\ x_S>\ =\ <+1\ ;\ -1>}$$

and it follows that it is an outright riskless, profitable and self-financing portfolio, by construction. ¶¶

Remark:

In case asset A my be overvalued, the lemma holds eventually by devising an arbitrage portfolio

$$\mathbf{\Delta P\ =\ <\ x_A\ ;\ x_S>\ =\ <-1\ ;\ +1>}$$

This outcome simplifies the Lemma 2 by using only one separation portfolio. Could we state something interesting about the ensuing arbitrage portfolio? It is for next lemma to frame a positive answer.

Lemma 4**: **Let ΔP be the arbitrage portfolio

$$\Delta P\ =\ <\ x_A\ ;\ x_S>\ =\ <+1\ ;\ -1>,$$

with S a separation portfolio with the same β as the mispriced asset A. Then, the total risk of the arbitrage portfolio equals the non-systematic risk of portfolio A.

$$\sigma(\Delta P)\ =\ \sigma_A(\varepsilon)$$

Proof: Recalling that the total risk of the arbitrage portfolio comes from

$$\mathbf{\sigma^2(\Delta P)\ =\ E\{\ R(\Delta P)\ -E[\ R(\Delta P)\ \}^2}$$

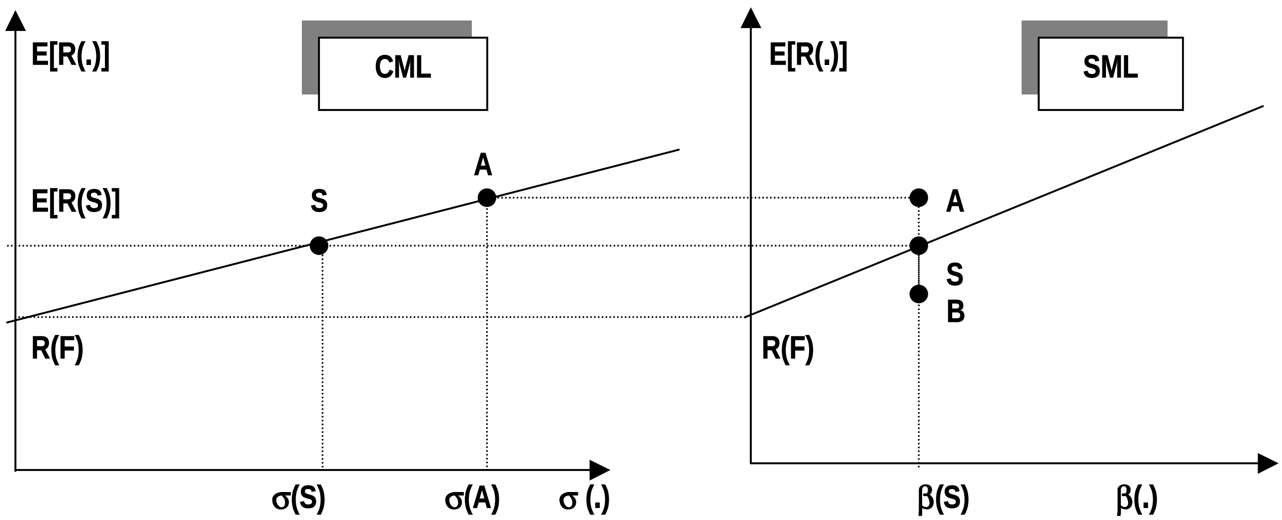
and expanding,

$$\sigma^2(\Delta P) = E\{ x_A . R(A) + x_S . R(S) - x_A . E[R(A)] - x_S . E[R(S)] \}^2$$

and

$$\sigma^2(\Delta P) = E\{ x_A . (R(A) - E[R(A)]) + x_S . (R(S) - E[R(S)]) \}^2$$

Figure 4: S has less total risk than A



Remark: *The arbitrage portfolio is :* $\Delta P = < x_A ; x_S > = < +1 ; -1 >$, *whereas its total risk comes as :* $\sigma(\Delta P) = \sigma_A^{\,2}(\varepsilon)$

that yields

$$\sigma^2(\Delta P) = x_A^{\,2} . \sigma_A^{\,2} + 2\, x_A . x_S . \sigma(R(A), R(S)) + x_S^{\,2} . \sigma_A^{\,2}$$

but the fractional parts in any arbitrage portfolios equal one in absolute value, so:

$$\sigma^2(\Delta P) = \sigma_A^{\,2} - 2\, \sigma(R(A), R(S)) + \sigma_S^{\,2}$$

and applying well known relationships (see the Appendix) and the fact that **A** and **S** have the same beta:

$$\sigma^2(\Delta P) = [\ \beta_A{}^2 . \sigma_M{}^2 + \sigma_A{}^2(\varepsilon)\ ] - 2\ \beta_A . \beta_A . \sigma_M{}^2 + \beta_A{}^2 . \sigma_M{}^2$$

Therefore:

$$\sigma^2(\Delta P) = \sigma_A{}^2(\varepsilon)$$

In other words, **ΔP** only inherits the specific risk of the asset **A**. ¶¶

Remark:

In case **A** might be overvalued, and thus located below the SML, for instance where **B** is shown in picture 4, the lemma holds eventually but the non-systematic risk component is still less than when **A** was undervalued. Therefore, the total risk in any arbitrage portfolio is greater when longing the mispriced asset than when we shorten it.

The outcome this lemma provides with has some bearing for portfolio managers in practice, because the total risk to keep under watch is translated only by the non-systematic component of the total risk carried out by the mispriced asset. Besides, as separation portfolios in real contexts are made out of concurrent positions in a market portfolio tracked by an index, and risk-free asset, such a simple structure helps the manager to cope with truly attainable portfolios.

CONCLUSIONS

This paper developed two alternative procedures to take advantage of the SML as a benchmark, but instead of taking any portfolio lying in the SML that shares the same beta with the mispriced asset, we firstly choose two separation portfolios to prove that such portfolios are feasible and that they really convey less total risk than any other pair of portfolios with the same address in the SML. Secondly, we did the same with only one separation portfolio. In this latter case, it was also proved that the total risk of an arbitrage portfolio consisting of a mispriced asset and a single separation portfolio only inherits the non-systematic risk of the mispriced asset.

APPENDIX

In this Appendix we labor on some relationships that were taken for granted in section 2. (Elton-Gruber (1995) and Blake (1999) provide broader background for these issues).

Let **S** be a separation portfolio

$$\mathbf{SB} = < x_F ; \; x_M >$$

Where $\mathbf{x}_F$ is the fraction of wealth invested in a risk-free asset, and $\mathbf{x}_M$ the remaining fraction allocated in the market portfolio.

The expected return of **S** comes from:

(A1) $$E[\,R(S)\,] = x_F \,.\, R(F) + x_M \,.\, E[\,R(M)\,]$$

while its variance follows from:

(A2) $$\sigma^2 (S) = x_F^{\,2} \,.\, \sigma_F^{\,2} + 2\, x_F \,.\, x_M \,.\, \sigma(\,R(F), R(M)\,) + x_M^{\,2} \,.\, \sigma_M^{\,2}$$

and this expression is simplifiable to

$$\sigma^2 (S) = x_M^{\,2} \,.\, \sigma_M^{\,2}$$

because of the distinctive features of any risk-free asset. Equivalently,

(A3) $$\sigma(S) = x_M \,.\, \sigma_M$$

that yields:

(A4) $$x_M = \sigma(S) \,/\, \sigma_M$$

Replacing (A4) in (A1), and profitting from the budget constraint

$$x_F + x_M = 1$$

after some calculations, we are led to the Capital Market Line:

$$E[\,R(S)\,] = R(F) + < \{\, E[\,R(M) - R(F)\,\} \,/\, \sigma_M > .\, \sigma_S$$

Next, as **S** lies in the SML, then its beta derives from:

$$\beta_S = x_F \,.\, \beta_F + x_M \,.\, \beta_M$$

but the risk-free asset has a zero beta, and the market portfolio has a beta equal to one. Then:

(A5) $$\beta_S = x_M$$

Also, profiting from A(4) and (A5):

(A6) $$\sigma(S) = \beta_S \cdot \sigma_M$$

Finally, from the market model we get:

$$\sigma^2(S) = \beta_S^{\ 2} \cdot \sigma_M^{\ 2} + \sigma_S^{\ 2}(\varepsilon)$$

it follows that:

$$\sigma_S^{\ 2}(\varepsilon) = 0$$

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