Online Appendix

To

Plugged in brokers:

A model about vote-buying and access to resources

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This appendix presents a series of technical definitions and proofs which support the argument that voters' welfare improves with their brokers skills at accessing resources. In particular, it presents: (i) a technical definition for players strategies, (ii) formal proofs showing that *B*´s optimal offer is unique, and (iii) a formal proof showing a voter's welfare improves with the skill of her broker at accessing resources.

Strategies

A behavioral strategy for B specifies his offer $z_t \epsilon [0, \pi_t]$ to the voter at each period as a function of the history preceding that period, and whether he received $\overline{\pi}$ or $\underline{\pi}$. In other words, a behavioral strategy for B is a set of offers z_t for all possible information sets defined by the previous history of offers and replies h_{t-1} and by the size of the pie available at time $t, \pi_t \epsilon \{\underline{\pi}, \overline{\pi}\}$. The previous history is given by all the actions the players have taken in previous rounds; it is the list of offers the broker(s) made from round 0 through round $t-1$: $\{z_j\}_{j=0}^{j=t-1}$, and all the replies the voter provided to these offers $C_j \in \{0,1\}_{j=0}^{j=t-1}$, where 0 is reject and 1 is accept. Any particular history before time t is then given by $h_{t-1} = \{(z_j, C_j)\}_{j=0}^{j=t-1}$, where I express for each round first the brokers' offer and then the voter's reply, and where the subscript indicates the round in which they are playing. I denote H_{t-1} as the set of all possible histories h_{t-1}. The set of strategies for B for any time t is then given by $\{z_t\}_{t=0}^{\infty} : H_{t-1} \times \pi_t \rightarrow [0, \pi_t]$.

A behavioral strategy for the voter has to specify the voter's reply to each possible offer she could receive from B in any round, given her type and previous history. The voter's strategy is a sequence of acceptance functions that specify the probability $K_t(z\,|h_{t-1}) \,\epsilon\, [0,1]$ that she will accept an offer z at any information set. Where, for example, $K_t(z\,|h_{t-1}) = 1$ means the voter accepts B's offer for sure, $K_t(z\,|h_{t-1}) = .5$ means the voter accepts with 50 percent probability, and $K_t(z\,|h_{t-1}) = 0$ means the voter rejects for sure. Therefore, a strategy for the voter is a mapping to an acceptance probability from all possible information sets defined by her type, previous history h_{t-1}, and given the offer she receives from B at time t; $z_t \epsilon [0, \pi_t]$. A sequence of behavioral strategies for the voter at each time can thus be formally expressed as $\{K_t\}_{t=0}^{\infty} : \varepsilon_t \times H_{t-1} \times z_t \rightarrow [0,1]$

Equilibrium

LEMMA 1. Only the negative solution (that I denote by $\tilde{z}$) to the square root that solves for z is feasible.

PROOF. Two roots solve for z (Equation 2);

$$z = r - u + \frac{2u}{\alpha\delta} \pm \frac{1}{\alpha\delta}\sqrt{2}u\sqrt{\frac{1}{u}\left(2u + r\alpha\delta - \underline{\pi}\alpha\delta^2 + \underline{\pi}\alpha^2\delta^2 - u\alpha\delta - \overline{\pi}\alpha^2\delta^2\right)}.$$

Note that since $\alpha\delta < 1$, then $((2u) / (\alpha\delta)) > 2u$. Therefore, the solution taking the positive root is greater than $r + u$. Since B would never make an offer greater than what is required to buy the highest voter type $(r + u)$, the positive root is out of the feasible offers. However, this positive root could possibly determine a corner solution at $r + u$. This is not the case because the asymptote, determined by the denominator of the expression that defines $P_b(z)$ is at $z = r - u + 2u/\alpha\delta$ (see that in Equation 2 if $z^* = r - u + 2u/\alpha\delta$ then the denominator is equal to 0), this is at a larger value of the horizontal axis than $r + u$ and at a smaller value than the positive solution to the square root. This establishes that the entire side of the equation at the right side of the asymptote does not cross the point $r + u$; it is entirely to the right of it, and, therefore, a solution at the positive root is out of the feasible values and cannot indicate a corner solution ■

LEMMA 2. The negative solution to the square root (denoted by $\tilde{z}$) is a maximum.

PROOF. The SOC of $P_b(z)$ with respect to z is given by $\frac{\partial^2 P_b(z)}{\partial^2 z} = \frac{4u\left(\alpha^2\delta^2(\overline{\pi}-\underline{\pi})-2u+\alpha\delta(\delta\underline{\pi}-r+u)\right)}{(2u+\alpha\delta(r-u-z))^3}$. The numerator is negative. The factor multiplying $4u$ in the numerator is the negative of the discriminant above. Since the discriminant must be positive, its opposite must be negative. The denominator is positive for $\tilde{z}$. Hence, the second derivative of $P_b(z)$ with respect to z is negative at $\tilde{z}$ and $\tilde{z}$ is a maximum ■

PROPOSITION 1. Upon receiving $\overline{\pi}$, B's optimal offer to the voter is unique.

PROOF. $P_b(z)$, as defined by Equation 2, is the broker's payoff when he is deciding what to offer given that he received the big pie $\overline{\pi}$. B wants to maximize $P_b(z)$ with respect to his offer z. The partial derivative of $P_b(z)$ with respect to z is

$$\frac{\partial P_b(z)}{\partial z} = \frac{2u\left(\delta\underline{\pi} + r - u - 2z\right) + \alpha\delta\left(r^2 + 2\overline{\pi}u - 2\underline{\pi}u + u^2 + 2uz + z^2 - 2r(u+z)\right)}{\left(2u + \alpha\delta(r - u - z)\right)^2}$$

Note that the denominator is always positive. Hence, the sign of the partial derivative of $P_b(z)$ with respect to z is determined by the sign of its numerator. By collecting terms in the numerator we can see that it is quadratic in z with positive sign:

$$\alpha\delta z^2 + z\left(2\alpha\delta u - 2\alpha\delta r - 4u\right) + \alpha\delta r^2 + 2\alpha\delta\overline{\pi}u + 2\delta\underline{\pi}u - 2\alpha\delta\underline{\pi}u + 2ru - 2\alpha\delta ru - 2u^2 + \alpha\delta u^2.$$

Therefore, the numerator is a convex parabola and has a minimum value. There are three different cases, according to where this minimum is, that need to be considered: (1) If this minimum is positive then the derivative of $P_b(z)$ with respect to z is always positive. This means that $P_b(z)$ is always increasing in z and that it achieves its maximum at the upper bound $r + u$. In this case the solution to the optimal solution is unique and given by the corner solution $z^* = r + u$. (2) If the minimum equals zero, then the quadratic expression of the numerator has only one root. Note that the derivative of $P_b(z)$ with respect to z is zero at that root and that it is positive everywhere else. Because in this case there is only one root, it must be that the discriminant of the quadratic formula is zero. From Equation 4 it is clear then that the optimal occurs at $z^* = r - u + 2u/\alpha\delta$. However, the asymptote determined by the denominator of the expression that defines $P_b(z)$ is exactly zero at $z^* = r - u + 2u/\alpha\delta$ (see that in Equation 2 if $z^* = r - u + 2u/\alpha\delta$ then the denominator is equal to 0), establishing that the derivative of $P_b(z)$ with respect to z is increasing for $z < r - u + 2u/\alpha\delta$, and hence the maximum is again at the corner solution $z^* = r + u$. (3) Finally, I consider the case in which the minimum is below zero and there are two distinct real roots solving for $\partial P_b(z)/\partial z$. These are the roots given by Equation 2. For the roots to be real and distinct in this case, the discriminant must be positive, so

$\left(\alpha^2\delta^2\left(-\overline{\pi}+\underline{\pi}\right)+2u-\alpha\delta\left(\delta\underline{\pi}-r+u\right)\right)>0$.If $P_b\left(z\right)$ is strictly concave it will have a unique maximum. $P_b\left(z\right)$ is strictly concave if its second derivative with respect to z is negative. I show next that this is the case. The second derivative of $P_b\left(z\right)$ with respect to z is given by $\frac{\partial P_b(z)}{\partial z}=\frac{4u\left(\alpha^2\delta^2(\overline{\pi}-\underline{\pi})-2u+\alpha\delta(\delta\underline{\pi}-r+u)\right)}{(2u+\alpha\delta(r-u-z))^3}$.The denominator is positive for any value of z smaller than the asymptote; $z<r-u+2u/\alpha\delta$. The factor multiplying $4u$ in the numerator is the negative of the discriminant above. Since the discriminant must be positive, its opposite must be negative. Hence, the second derivative of $P_b\left(z\right)$ with respect to z is negative and $z^*=\tilde{z}$. Therefore, when B tries to buy the voter there is a unique optimal offer z^* ■

Brokers' access to resources and voters' wellbeing

Note that since the offer z^* would not increase further if equal to $r+u$, and that the voter rejects the offer if $z^*<r-u$, I do the partial derivatives with respect to the optimal offer z^* when the optimal offer is interior; that is, when $z^*=\tilde{z}$.

Proposition 2. The optimal offer z^* is increasing in α.

Proof. I prove next that the partial derivative of z^* with respect to α is always positive; $\partial z^*/\partial\alpha>0$. The derivative of $\tilde{z}$ with respect to α is given by

$$\frac{\partial\tilde{z}}{\partial\alpha}=\frac{u\left(4\sqrt{2u}-\sqrt{2}\alpha\delta\left(\delta\underline{\pi}-r+u\right)-4\sqrt{u\left(2u+r\alpha\delta-\underline{\pi}\alpha\delta^2+\underline{\pi}\alpha^2\delta^2-u\alpha\delta-\overline{\pi}\alpha^2\delta^2\right)}\right)}{2a^2d\sqrt{u\left(2u+r\alpha\delta-\underline{\pi}\alpha\delta^2+\underline{\pi}\alpha^2\delta^2-u\alpha\delta-\overline{\pi}\alpha^2\delta^2\right)}}.$$

Note that the denominator is always positive. Therefore, we need the numerator to be positive too. Note that if the numerator is positive for the highest possible value of the discriminant it will always be positive. If the offer is interior, it has to be that $\tilde{z}>r-u$, which implies—from Equation 4—that $\left(\sqrt{2}\sqrt{u\left(2u+r\alpha\delta-\underline{\pi}\alpha\delta^2+\underline{\pi}\alpha^2\delta^2-u\alpha\delta-\overline{\pi}\alpha^2\delta^2\right)}\right)/\left(\alpha\delta\right)<2u/\left(\alpha\delta\right)$. If not $\tilde{z}>r-u$. Setting this expression equal to its greatest possible value yields $\sqrt{u\left(2u+r\alpha\delta-\underline{\pi}\alpha\delta^2+\underline{\pi}\alpha^2\delta^2-u\alpha\delta-\overline{\pi}\alpha^2\delta^2\right)}=\sqrt{2}u$. Note now that by replacing the root in the numerator with $\sqrt{2}u$ we get $u\left(4\sqrt{2u}-\sqrt{2}\alpha\delta\left(\delta\underline{\pi}-r+u\right)-4\sqrt{2u}\right)$, which simplifies to: $-u\sqrt{2}\alpha\delta\left(\delta\underline{\pi}-r+u\right)$. Since $\delta\underline{\pi} < r-u$, then $\delta\underline{\pi}-r+u < 0$. Therefore, $-u\sqrt{2}\alpha\delta\left(\delta\underline{\pi}-r+u\right)$ is always positive, which means that the numerator is positive and that $\partial z^*/\partial\alpha > 0$. ■

PROPOSITION 3. The optimal offer z^* is increasing in $\overline{\pi}$.

PROOF. The partial derivative of z^* with respect to $\overline{\pi}$ when $z^* = \tilde{z}$ is positive;

$$\frac{\partial\tilde{z}}{\partial\overline{\pi}} = \frac{\alpha\delta u}{\sqrt{2}\sqrt{u\left(2u+r\alpha\delta-\underline{\pi}\alpha\delta^2+\underline{\pi}\alpha^2\delta^2-u\alpha\delta-\overline{\pi}\alpha^2\delta^2\right)}} \quad ■$$

PROPOSITION 4. The voter's payoff in equilibrium is increasing in α.

PROOF. I denote by $U_v\left(z^*\right)$ the ex-ante expected equilibrium payoff for the voter (prior to knowing her type), and by $U_v\left(z^*,\varepsilon\right)$ the equilibrium payoff for a voter that already knows her type. Note that since brokers are all identical, the voter can expect all the brokers to make the same offer. This means that depending on the size of the pie, the optimal offer is always the same. The ex-post payoff for a voter (upon learning her type ε) is given by

$$U_v\left(z^*,\varepsilon\right) = \begin{cases} r+\varepsilon+\delta U_v\left(z^*,\varepsilon\right) & \text{if } \varepsilon > z^*-r \\ \left(1-\alpha\right)\left(r+\varepsilon+\delta U_v\left(z^*\right)\right)+\alpha\left(z+\delta U_v\left(z^*\right),\varepsilon\right) & \text{if } \varepsilon \leq z^*-r \end{cases}$$

where the first line captures the payoff for a voter that will reject the offer z^*, and the second line captures the payoff for a voter that will accept the offer z^*. Note now that the payoff in equilibrium for a voter at the beginning of the game before she knows her reservation value is equal to the expectations of $U_v\left(z^*,\varepsilon\right)$ over realizations of ε. Under the assumption of uniformity of ε this is formally $U_v\left(z^*\right) = \int_{-u}^{u} U_v\left(z^*,\varepsilon\right) d\frac{\varepsilon}{2}$. Therefore,

$$U_v\left(z^*\right) = \int_{\hat{\varepsilon}}^{u} r+\varepsilon+\delta U_v\left(z^*\right) d\frac{\varepsilon}{2u} + \int_{-u}^{\hat{\varepsilon}} \left(1-\alpha\right)\left(r+\varepsilon+\delta U_v\left(z^*\right)\right)+\alpha\left(z^*+\delta U_v\left(z^*\right)\right) d\frac{\varepsilon}{2u},$$

where the first integral captures the payoff for types of the voter that will reject the offer z^*, the second integral captures the payoff for types that will accept the offer z^*, and where $\hat{\varepsilon}$ is the indifferent voter to the offer z^* such that $\hat{\varepsilon} = z-r$. Solving the integrals and then solving for $U_v(z^*)$ yields, $U_v(z^*) = \frac{1}{1-\delta}\left(r + \frac{\alpha(z^*-r+u)^2}{4u}\right)$.So now we want to prove that this payoff $U_v(z^*)$ is increasing in α. Because$\frac{dU_v(z^*)}{d\alpha} = \frac{\partial U_v(z^*)}{\partial z^*}\frac{\partial z^*}{\partial \alpha} + \frac{\partial U_v(z^*)}{\partial \alpha}$, it suffices to prove that $\frac{\partial U_v(z^*)}{\partial z^*}\frac{\partial z^*}{\partial \alpha} + \frac{\partial U_v(z^*)}{\partial \alpha} > 0$, *where*

$$z^* = \frac{1}{\alpha\delta}\left(2u + r\alpha\delta - u\alpha\delta - \sqrt{2}u\sqrt{\frac{1}{u}\left(2u + r\alpha\delta - \underline{\pi}\alpha\delta^2 + \underline{\pi}\alpha^2\delta^2 - u\alpha\delta - \overline{\pi}\alpha^2\delta^2\right)}\right).$$

The partial derivative of $U_v(z^*)$ with respect to z^* is equal to $\alpha(z^* - r + u)/2u(1-\delta)$. Since the solution is interior it must be that $z^* \geq r - u$, thus $\partial U_v(z^*)/\partial z^* \geq 0$. It has already been proved in Proposition 2 that $\partial z^*/\partial\alpha > 0$. Finally, the partial derivative of $U_v(z^*)$ with respect to α is equal to $(z - r + u)^2/(4u(1-\delta))$, which is positive. Therefore, $\partial U_v(z^*)/\partial z^* \geq 0; \partial z^*/\partial\alpha > 0$ and $\partial U_v(z^*)/\partial\alpha > 0$ implies that, $dU_v(z^*)/d\alpha > 0$ ■