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IMPOSING CONCAVITY AND THE NULL-JOINTNESS PROPERTY ON THE PRODUCTION POSSIBILITIES FRONTIER IN CASE OF POLLUTING TECHNOLOGIES

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Economic theory requires the directional distance functions used to study the properties of production possibility sets of polluting technologies to be concave in both outputs, while the implied production possibilities frontier (PPF) is required to be concave with respect to the bad output. However, existing estimation frameworks do not preclude the estimation of convex PPFs. We analyze geometrical properties of the quadratic approximation to the directional output distance functions to derive a constraint that guarantees PPF concavity and consider the issue of imposing the property of null-jointness on the production possibilities set, which is also required by theory. We simulate a dataset corresponding to a concave PPF and show that in case concavity and null-jointness constraints are not imposed, it is possible that the conventional estimation framework may lead to erroneous conclusions with respect to the type of curvature of both the directional output distance function, and the PPF.

JEL classification codes: D24, Q53, O44, R11
Key words: CO_2 emissions, marginal abatement cost, distance function

I. Introduction

It is convenient to model polluting production processes in terms of the multi-output technologies with at least one output, e.g., CO_2 emissions, being an undesirable bad. Output distance functions suggested by Shephard (1970) are a useful analytical tool allowing one to quantify such technologies without having to aggregate multiple outputs into a single output index. In its essence, the output distance function is

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representing the distance between observed output combinations, and their efficient projections on the production possibilities frontier (PPF) with greater values of the distance function corresponding to less productive efficiency. Distance to the PPF in this context is measured along a vector emanating from the observed output mix, for example a vector (1,1) corresponding to a simultaneous increase in the amounts of both outputs.

Directional output distance functions developed by Fare et al. (2005) are well-suited for the analysis of polluting technologies since the directional vector along which the efficient projection is computed implies an increase in the good output(s), as is the case with the conventional output distance function, along with the *reduction* in the bad output, which corresponds to the idea of reducing pollution levels. In this study we deal with the two-output polluting technology with one good output y, such as GDP, and one bad output b, such as the CO2 emissions levels.

Given the global importance of reducing pollution levels, computing the costs of such reduction is a necessary task. One of the more popular approaches undertaken in the literature is to exploit the duality between the output distance function and the revenue function to compute the shadow price of an undesirable output as a slope of the PPF at the efficient projection point for each observed combination of outputs, often referred to as the marginal abatement costs (MAC) of reducing the bad output.

Most empirical studies that estimate output distance functions, directional or not, employ a quadratic approximation to the true distance function whose parameters are estimated by minimizing the sum of individual values of the distance function subject to a number of constraints that reflect the desired properties of the underlying production possibilities set (PPS). The translation property constraint, for instance, makes sure that the estimated quadratic function in two outputs actually has the distance function properties, i.e., it turns into zero if the good output is increased, and the bad output is decreased by the value of the approximating quadratic function times the corresponding component of the directional vector. Monotonicity constraints require the (directional) output distance function to increase in the bad output, and to decrease in the good one. If monotonicity constraints hold, the resulting PPF is upward sloping, reflecting the desirable property of the positive MACs.

An important theoretical property of the PPS of the polluting technology is that its production possibilities frontier, viewed as a function $y = f^{PPF}(b)$ of the level of bad output, has to be concave, i.e., $y''(b) \leq 0$, see, e.g., Fare et al. (1993). In addition, the distance function should inherit the properties of the underlying PPS by being a concave function of both outputs. In general, concavity of a quadratic function does not necessarily imply the concavity of any of its level curves, including the PPF defined by $D(\bullet)=0$, where $D(\bullet)$ stands for a distance function. In this study, we formally prove that due to the translation property imposed on the quadratic approximation to the directional output distance function, concavity of the latter in both outputs implies the PPF concavity, and formally derive the concavity constraints on the PPF and the distance function. We demonstrate that it is the translation property imposed on the parameters of the estimated approximation curves that constrains them to be either parabolas, or straight lines. In the latter case, the PPF concavity is satisfied automatically. However, in the former case the PPF may be approximated by convex parabolas, but we suggest that this can be avoided by imposing concavity constraints.

It is somewhat surprising that the overwhelming majority of existing empirical studies limit the set of constraints to the ones that guarantee positivity of the MACs and the fulfillment of the translation property, along with a few technical constraints. We provide an empirical example of a simulated polluting technology characterized by a concave production possibilities frontier that is estimated to be convex by applying the conventional directional distance function approach. In this way we are demonstrating the danger of not imposing concavity constraints on the estimated parameters of the directional output distance function and, as a consequence, on the PPF.

The empirical study that is the closest to ours appears to be O'Donnell and Coelli (2005) where the authors impose concavity on the PPF by employing a Bayesian estimation framework. Ours is a much simpler approach to resolving the issue of PPF concavity. We also consider the imposition of the null-jointness property on the implied production possibilities set, which precludes technologies allowing "clean" production of a good output, i.e., the possibility of producing positive amounts of a good output with no pollution at all. The null-jointness constraints are not normally imposed in the empirical literature either, which is why in addition to the concavity constraints, we suggest imposing a necessary condition for the null-jointness property.

Here, we strongly argue against interpreting the individual approximating curves as the individual PPFs for each observation since it is only a small part of the (directional) distance function's range, namely the vicinity of the estimated efficient projection point on the PPF that is used as an approximation to the true PPF. The rest of the approximating curve is unrelated to the true PPF, and as such cannot be used for inference on its properties. This implies, for instance, that the null-jointness property can only be imposed locally, i.e., at the observed combinations of outputs, rather than in terms of the parameters of the true PPF that are unknown.

This paper is organized as follows. In the next section we discuss the theoretical framework underlying the directional output distance function approach to the estimation of marginal abatement costs. In Section III we discuss the geometrical properties of the quadratic approximation to the distance function, and derive the constraints ensuring the directional output distance function and PPF concavity, and the necessary null-jointness constraints. In Section IV we discuss our empirical results. Section V concludes.

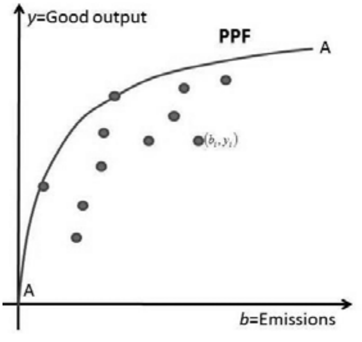
II. Theoretical framework

A. Modeling polluting technologies

Consider $P(\vec{x})=\{(b,y):\vec{x}\text{ can produce }(b,y)\}$ be the set of all output vectors (b,y) that can be produced using input vector $\vec{x}$ where *b* and *y* are the amounts of bad and good output, respectively. Assume all inputs are strongly disposable: if $\vec{x}'\geq\vec{x}$ then $P(\vec{x})\subseteq P(\vec{x}')$, i.e., increasing the inputs from $\vec{x}$ to $\vec{x}'$ will keep the output mix produced with $\vec{x}$ feasible. Weak disposability assumption is imposed on both outputs, namely, $(b,y)\in P(\vec{x})\Rightarrow(\theta b,\theta y)\in P(\vec{x})$ for any real $\theta\in[0,1]$: it is possible to proportionally scale down the production of both outputs simultaneously. $P(\vec{x})$ is assumed to be compact to make sure infinite amounts of outputs may not be produced with the finite amounts of inputs.

The next two assumptions are specific to the modeling of production sets with bad outputs. Null jointness is imposed in order to preclude the possibility of pollution-free production, i.e., $(y,0)\in P(\vec{x})$ implies y=0. Strong disposability is assumed for the desirable outputs, but is denied in case of the undesirable ones. That is, if $(b,y)\in P(\vec{x})$ then $(b,y-\Delta y)\in P(\vec{x}),\Delta y\geq 0$, e.g., it is always possible to reduce the amount of a good output by $\Delta y\geq 0$ while keeping that of a bad output intact, but the converse is not allowed, reflecting the idea of a costly reduction of the amount of bad output. The production possibilities frontier (PPF) function is defined as the maximum amount of good output *y* that can be produced given a particular amount of bad output *b*. The PPF function $y=y^{PPF}(b)$ is required to have a positive first-order derivative to reflect costly reduction of the bad output, and a non-negative second-order derivative. The latter property is based on the assumption of the non-decreasing marginal costs of pollution *reduction*. The property of null-jointness implies that the only way to produce any positive amount of good output *y* is to produce it along with some amount of the bad output b, i.e., $(b,y) \notin P(\vec{x}) \forall (b,y) \in \{(b,y): b=0, y>0\}$. The assumptions on the production set outlined above result in a specific shape of the PPF, illustrated in Figure 1 below.

Figure 1. PPF of the production possibilities set with good and bad output



In Figure 1, curve AA is the production possibilities frontier. The PPF is positively sloped, with the magnitude of the slope interpreted as the marginal abatement cost of reducing bad output in terms of the good one. The positively-sloped PPF is the one feature that creates the difference with the PPF shape in case both outputs are good ones, reflecting the fact that, if producing efficiently, producing more of the good output entails creating more pollution. Marginal abatement costs are non-increasing in the amount of bad output, which is equivalent to saying that marginal costs of pollution reduction are non-decreasing in the reduction volume. Alternatively, higher production levels of good output involve creating increasingly more pollution. If that were not the case, i.e., if the PPF as a function of bad output were convex as a function of bad output, rather than concave, we would not have to deal with the problem of increasing environmental damage since after a certain threshold level of good output has been transcended, further increases in good output would only involve minor growth in pollution levels. For that reason, it is crucial that the estimated production set be convex, i.e., that the PPF as a function of bad output be concave.

B. Directional output distance functions

Fare et al. (2005) provided an operational way of parameterizing the production possibilities set described above by introducing the concept of a directional output distance function based on the idea by Shephard (1970) of the output distance function that can be shown to be dual to the revenue function. The key idea is to estimate a quadratic approximation of the directional distance function that maps actually observed output combinations (b_i,y_i) to the distances between (b_i,y_i) and the true PPF. Those distances are measured along a directional vector (hence the name) $(-g_b,g_y),g_b>0,g_y>0$, which we normalize to be a unit vector, i.e., $\sqrt{g_b^2+g_y^2}=1$. The directional distance function $\vec{D}_i\left(b_i,y_i\middle|\vec{x}_i,g_b,g_y\right)$ is defined as follows:

$$\vec{D}_i\left(b_i,y_i\middle|\vec{x}_i,-g_b,g_y\right)=\max\left\{\tau:\left(b_i-\tau g_b,y_i+\tau g_y\right)\in P\left(\vec{x}_i\right)\right\}, \tag{1}$$

where $\vec{x}_i$ is an input vector employed by production unit i, and the vector sign in $\vec{D}_i$ emphasizes the fact that (1) is representing a *directional* output distance function as opposed to just an output distance function. The subscript i in $\vec{D}_i(\bullet)$ above is to emphasize the fact that parameters of the relationship implicitly defined by $\vec{D}_i(\bullet)=0$ will be different depending on the input vector $\vec{x}_i$ so that an implicit function $y = y(b)$ defined by $\vec{D}_i(\bullet)=0$ in the vicinity of a particular observation (b_i,y_i), in case this implicit function exists in that vicinity, will have different parameters compared to the similar implicit function in the vicinity of (b_j,y_j), $j \neq i$.

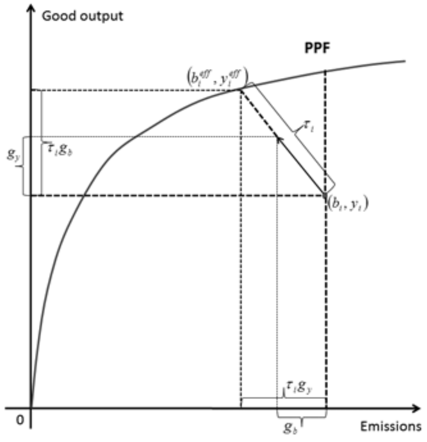
Denoting τ_i to be the solution of the maximization problem in the right-hand side of (1), the directional distance function can be interpreted to project the (inefficient) output mix (b_i,y_i) to its efficient projection $\left(b_i^{eff},y_i^{eff}\right)$, which is a point on the PPF that pointed at by the directional vector $(-g_b,g_y)$ that emanates from the observed (b_i,y_i), and where $b_i^{eff}=b_i-\tau_i g_b, y_i^{eff}=y_i+\tau_i g_y$ (see Figure 2 below).

Consider a movement from the observed output mix (b_i,y_i) towards the PPF along the directional vector $(-g_b,g_y)$ by simultaneously decreasing bad output by τg_b, while increasing the amount of good output by τg_y, where $\tau\geq 0$. Geometrically that means that we will get closer to the PPF by the distance of $\sqrt{\tau^2 g_b^2+\tau^2 g_y^2}=\tau$ since the directional vector $(-g_b,g_y)$ is assumed to be normalized to unity. By definition of the directional distance function, the latter then must satisfy the translation property:

$$\vec{D}_i\left(b_i,y_i\middle|\vec{x}_i,g_b,g_y\right)-\vec{D}_i\left(b_i-\tau g_b,y_i+\tau g_y\middle|\vec{x}_i,g_b,g_y\right)=\tau,\forall\tau\geq 0,i=1,...T, \tag{2}$$

where T is the number of observations. Since negative τ's correspond to the directional vector collinear with $(-g_b,g_y)$, but looking in the opposite direction, the directional distance function will assume negative values for all output combinations that do not belong to the underlying production possibilities set. Figure 2 illustrates the idea of the directional output distance function.

Figure 2. Directional output distance function



C. Empirical estimation of directional output distance functions

It became common in the empirical literature to estimate a second-order approximation to the distance function (1) subject to a set of constraints ensuring the fulfillment of the translation property and the positive sign of the implied PPF slope, which is linked to the distance function as follows:

$$\left.\frac{\partial y}{\partial b}\right|_{(b_i-\tau_i g_b, y_i+\tau_i y_b)} = -\left.\frac{\partial \vec{D}_i \big/ \partial b}{\partial \vec{D}_i \big/ \partial y}\right|_{(b_i, y_i)}, \tag{3}$$

where $\vec{D}_i = \vec{D}(b_i, y_i | \vec{x}_i, g_b, g_y)$. This approach has been adopted in studies such as, e.g., Lee (2011), and Maradan and Vassiliev (2005). Equation (3) says that the negative of the ratio of the two partial derivatives of the distance function at the observed output mix (b_i,y_i) is equal to the PPF slope at the efficient projection of (b_i,y_i). This slope can be interpreted as the marginal abatement cost of reducing bad output by one unit incurred at (b_i,y_i). Fare et al. (2005) used duality theory to demonstrate that the right-hand side of (3) is equal to the ratio of the shadow prices of the two outputs.

A substantial number of empirical studies assuming the directional output distance function approach to the estimation of marginal abatement costs of reducing bad output employ the following empirical specification:

$$
\begin{aligned}
&Min\sum_{i=1}^{T}\vec{D}_i-0,\\
&\vec{D}_i=\alpha_0+\sum_{n=1}^{K}\alpha_n x_{in}+\beta_1 y_i+\gamma_1 b_i+\frac{1}{2}\sum_{n=1}^{K}\sum_{n'=1}^{K}\alpha_{nn'}x_{in}x_{in'}+\frac{1}{2}\beta_2 y_i^2+\frac{1}{2}\gamma_2 b_i^2+\sum_{n=1}^{K}\nu_n x_{in}b_i+\sum_{n=1}^{K}\delta_n x_{in}y_i+\mu b_i y_i,\\
&\vec{D}_i\geq 0,\frac{\partial\vec{D}_i}{\partial y}\leq 0,\frac{\partial\vec{D}_i}{\partial b}\geq 0,\vec{D}\left(\vec{x},b-\tau g_b,y+\tau g_y\middle|g_b,g_y\right)=\vec{D}\left(\vec{x},b,y\middle|g_b,g_y\right)-\tau,\tau\in\Re,\alpha_{nn'}=\alpha_{n'n},n=1..K,
\end{aligned}
\tag{4}
$$

where T is the number of observations and K is the number of inputs.

The first constraint requires that the observed output mixes belong to the production set, the next two constraints ensure the positive slope of the PPF at the observed output mix, the next to last constraint ensures the translation property is satisfied, and the last constraints are the technical constraints on the coefficients' symmetry. The constraints for the estimated directional output distance function to satisfy the translation property for an arbitrary directional output vector were derived in Fare et al. (2006). We reproduce them below for further reference:

$$
\begin{aligned}
&\beta_1 g_y-\gamma_1 g_b=-1,\\
&\delta_n g_y-\nu_n g_b=0,n=1,...,K,\\
&\beta_2 g_y=\mu g_b,\\
&\gamma_2 g_b=\mu g_y.
\end{aligned}
\tag{5}
$$

III. The geometry of second-order approximation to the directional output distance function

A. Quadratic approximation to the distance function as parabolas or straight lines

Solving the linear program (4) results in a quadratic function in K+2 variables, namely, the two outputs (b,y) and the K inputs $\left(x_1,...,x_K\right)=\vec{x}$, which implies that observed input vectors $\vec{x}_i$ for each observation determine a second-order curve $\vec{D}_i\left(b,y\middle|g_b,g_y,\vec{x}_i\right)=0$ in two outputs of the following form:

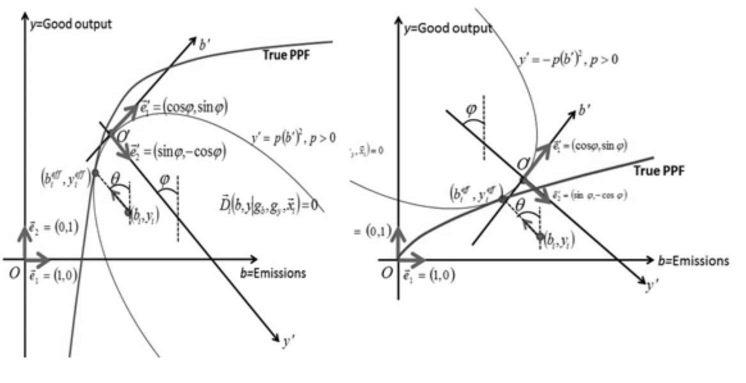
$$
\begin{aligned}
&\vec{D}_i\left(b,y\middle|g_b,g_y,\vec{x}_i\right)=\left(\frac{1}{2}\beta_2\right)y^2+\left(\frac{1}{2}\gamma_2\right)b^2+\mu by+\left(\gamma_1+\sum_{n=1}^{K}\nu_n x_{in}\right)b\\
&+\left(\beta_1+\sum_{n=1}^{K}\delta_n x_{in}\right)y+\left[\alpha_0+\sum_{n=1}^{K}\alpha_n x_{in}+\frac{1}{2}\sum_{n=1}^{K}\sum_{n'=1}^{K}\alpha_{nn'}x_{in}x_{in'}\right]=0,\\
&i=1,...,T.
\end{aligned}
\tag{6}
$$

The quadratic function (6) may define several second- and first-order curves in the space of two variables *b* and *y* such as, e.g., a parabola, an ellipse, a hyperbola, or a straight line. In the Online Appendix we prove the following proposition:

Proposition 1.
(a) Due to the translation property, for each input vector $\vec{x}_i$ *corresponding to the observed* (b_i,y_i) *the relationship* (6) *is implicitly defining a relationship between good output y and bad output b* $\vec{D}_i\left(b,y|g_b,g_y,\vec{x}_i\right)=0$ *that is either a parabola or a straight line.*
(b) In case $\vec{D}_i\left(b,y|g_b,g_y,\vec{x}_i\right)=0$ *is a parabola, the axis of symmetry of each suchparabola for all i* =1,…,*Tis forming the same angle φ with the vertical axis, i.e.,* $\varphi_i=\varphi_j \forall i,j=1,...,T$.
(c) The angle θ between the directional vector $(-g_b,g_y)$ *and the vertical axis is linked to the angle φ as follows:* $\theta=\frac{\pi}{2}-\varphi$.
(d) Concavity: For the approximated PPF at the projection point $\left(b_i^{eff},y_i^{eff}\right)$ *to be concave, it is sufficient to require for each observation that: sign*$(\mu)=-$*sign*$(\gamma_1 g_b-\beta_1 g_y)$.

Since in the linear case (i.e., the one when $\beta_2=\gamma_2=0$), concavity of the approximating curve $\vec{D}_i\left(b,y|g_b,g_y,\vec{x}_i\right)=0$ is guaranteed automatically, let us consider the parabolic case, i.e., the one for which $\beta_2\neq 0$ or $\gamma_2\neq 0$. In the Online Appendix we prove that due to the translation property the relationship (6) implicitly defines either a parabola of the form $y'=p(b')^2, p>0$ or a parabola $y'=-p(b')^2, p>0$,

Figure 3. Approximating parabolas: concave and convex cases



Note: Angles θ and j are linked by the relationship $\varphi=\frac{\pi}{2}-\theta$.

where (b',y') are the parabolas' coordinates in a new coordinate system, and p is a positive number that is a function of the distance function's parameters *and* the input levels for a particular observation. The new coordinate system is obtained from the original coordinate system by a parallel shift and rotation of the latter, as demonstrated by Figure 3 above. As we demonstrate below, the failure to impose the concavity constraints in part (d) of Proposition 1 may result in a convex approximation to the concave underlying PPF, which is undesirable on theoretical grounds.

B. Concavity of the directional output distance function and ppf concavity

It follows from the PPF concavity conditions in part (d) of Proposition 1 that the PPF concavity is equivalent to concavity of the directional output distance function in both outputs. Indeed, the distance function's Hessian at the observed combination of outputs is identically equal to zero for all possible directional vectors $(-g_b,g_y)$. The Hessian $H=\begin{vmatrix}\beta_2 & \mu \\ \mu & \gamma_2\end{vmatrix}=\beta_2\gamma_2-\mu^2\equiv 0$, since by translation property in (5), $\beta_2\gamma_2\equiv\mu^2$ irrespectively of the directional vector. It follows that the distance function will be concave (but not necessarily strictly concave) at all of the observed output combinations if $\beta_2\leq 0$ or, equivalently, if $\gamma_2\leq 0$. The latter two conditions are equivalent to the condition on PPF concavity since by the translation property, $\beta_1 g_y-\gamma_1 g_b=-1$, and thus the PPF concavity condition sign(μ) = –sign($\gamma_1 g_b-\beta_1 g_y$) reduces to sign(μ) = –sign(1), or $\mu\leq 0$. Since by the translation property the coefficients μ, β_2 and γ_2 are of the same sign, concavity of the directional distance function in both outputs implies the PPF concavity, and the other way round.

It is worthwhile noticing that in general, the concavity of a quadratic function does not guarantee the concavity of any of its level curves, of which the PPF is one. In case the polluting technology produces one good and one bad output, the translation property ensures the equivalence of concavity conditions imposed on the parameters of the directional output distance function, and the PPF. It is the subject of our further research to see whether the two concavity requirements remain equivalent in the more general case of more good and bad outputs.

C. Implications for inference on the true PPF parameters

Irrespectively of the estimated parameters, in general the quadratic function (6) is implicitly defining more than one curve linking b and y. This happens since the coefficients for b and y, i.e., the coefficients $\gamma_1 + \sum_{n=1}^{K} v_n x_{in}$ and $\beta_1 + \sum_{n=1}^{K} \delta_n x_{in}$, as well as the term $\alpha_0 + \sum_{n=1}^{K} \alpha_n x_{in} + \frac{1}{2} \sum_{n=1}^{K} \sum_{n'=1}^{K} \alpha_{nn'} x_{in} x_{in'}$, depend on the observation-specific values of production inputs $x_{i1}, x_{i2}, x_{i3}, i = 1..T$. Equation (6) in this way is defining a family of T curves, one for each observed combination of outputs (b_i, y_i).

It is important to notice that it is only the small part of the approximating parabola defined by (6) in the vicinity of the efficient projection $\left(b_i^{eff}, y_i^{eff}\right)$ that is approximating the true PPF parameters and whose slope at that projection is used for the estimation of MACs. Outside of this vicinity this parabola cannot be considered to be an efficient boundary of the production possibilities set for observation i. As a result, the estimated parameters of the directional output distance function in (6) cannot be used for inference on the true PPF parameters.

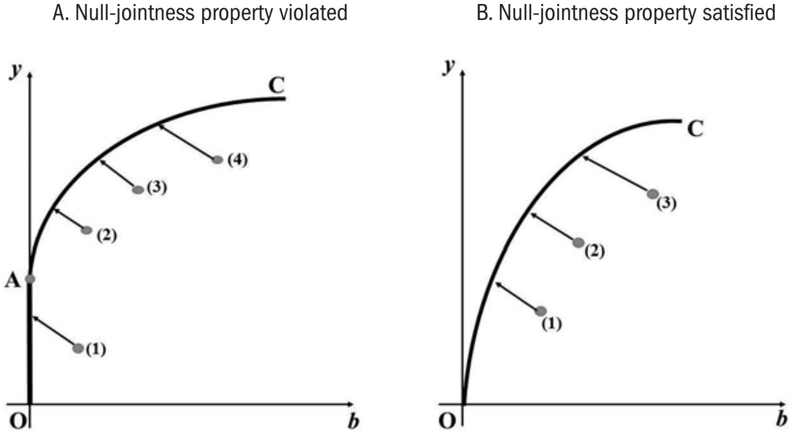
D. Implications for the null-jointness property

The null-jointness property is imposed on the PPS in order to preclude the possibility of producing a positive amount of good output while emitting zero pollution levels. Figure 4.A below depicts a PPS whose efficient frontier *OAC* does not satisfy the null-jointness property since it includes segment *OA*, which implies it is possible to produce positive levels of good output up to level *A* without emitting any pollution at all. The efficient frontier *OC* in Figure 4.B, on the other hand, satisfies the null-jointness property because pollution-free production of good output is impossible.

If it were possible to estimate the parameters of the directional distance function for all combinations (b,y), imposing the null-jointness property would boil down to requiring the value of the directional output distance function to be negative for all output combinations that satisfy $b = 0$ and $y = 0$. However, as discussed in the previous sub-section, the quadratic function in (6) is only approximating the true distance function in the close vicinity of $\left(b_i^{eff}, y_i^{eff}\right)$, which is not necessarily close to the locus of output combinations characterized by $b = 0$ and $y = 0$. Hence, the estimated parameters in (6) cannot be used to impose the null-jointness property.

Nevertheless, it appears possible to impose a *necessary* condition for the null-jointness property. Namely, as illustrated by Figure 4.B, if a PPS satisfies the null-jointness property, its efficient frontier must be such that all of the projections on it of the observed output combinations must satisfy $b_i^{eff} > 0$. This condition is not sufficient since, as can be inferred from Figure 4.A, if the sample of observations were limited to (2), (3) and (4), the efficient levels of bad output would be positive, while the PPS itself would not satisfy the null-jointness property. We suggest adding the set of constraints $b_i^{eff} > 0$ in addition to the concavity constraints as a second-best solution to the problem of ensuring the satisfaction of the null-jointness property.

Figure 4. Necessary condition for null-jointness property



IV. Empirical exercise

A. Simulation

Consider an industrial plant producing two outputs: a good output y and a polluting output b. Let there be three factors of production: capital k, labor l, and fuel f. Assuming a Cobb-Douglas production function for the good output and a linear production function for the bad output, the PPF $y^* = f(b)$ will be a solution to the following optimization problem:

$$\begin{aligned} &\underset{k,l,f}{Max}\, y = k^{\alpha} l^{\beta} f^{\gamma} \\ &s.t.\;\; \delta k + \varphi l + \psi f = b, \end{aligned} \tag{7}$$

where the Greek letters stand for the real positive-valued parameters. Maximizing the Lagrangian results in the following solutions for the optimal levels of capital, labor and fuel as functions of the two production functions' parameters and the pollution level b:

$$k^* = \frac{\alpha}{\delta}\frac{b}{\alpha+\beta+\gamma},$$
$$l^* = \frac{\beta}{\varphi}\frac{b}{\alpha+\beta+\gamma}, \tag{8}$$
$$f^* = \frac{\gamma}{\psi}\frac{b}{\alpha+\beta+\gamma}.$$

Substituting these optimal levels of factor inputs back into the Cobb-Douglas production function in (7), one obtains a parametric expression for the PPF:

$$y^* = \left(\frac{\alpha}{\delta}\right)^{\alpha}\left(\frac{\beta}{\varphi}\right)^{\beta}\left(\frac{\gamma}{\psi}\right)^{\gamma}\frac{b^{\alpha+\beta+\gamma}}{(\alpha+\beta+\gamma)^{\alpha+\beta+\gamma}} = Z \times b^{\alpha+\beta+\gamma}, \tag{9}$$

where Z is a positive real number. The PPF in (9) will be concave if $\alpha+\beta+\gamma<1$.

In performing our simulation exercise, we set $\alpha=\beta=\gamma=0.25$, and $\delta=\varphi=\psi=0.33$, which according to (9) results in the PPF function $y^* = f(b) \approx 1.01 \times b^{0.75}$ that is obviously concave in b. We generate a sample of uniformly distributed values of bad output $b \in (0,1)$ and use (8) and (9) to compute optimal values of the three production factors and the PPF value y^*. To introduce random variation into the dataset, we added a random noise to k^*, l^*, f^* and y^*. Table 1 below contains summary statistics for the simulated dataset.

In Table 2 below we present the results of our estimations of the parameter μ in the directional output distance function specification (6) whose sign determines the type of the approximating parabola. The estimation was done by linear programming in the SAS environment. In the column "Benchmark Model" we report estimation results of the model in (4) without imposing concavity and null-jointness constraints. In the next column we report results for a model with these

Table 1. Summary statistics for the simulated dataset

	Mean	Standard deviation	Minimum	Maximum
Good output, y	1.07	0.31	0.32	1.56
Bad output, b	0.55	0.29	0.003	0.99
Capital, k	0.78	0.34	0.08	1.41
Labor, l	0.81	0.35	0.04	1.44
Fuel, f	0.81	0.34	0.11	1.45

Note: the number of observations is 77, the simulation was performed in Excel using the uniform random number generator function rand().

Table 2. Concavity of the estimated distance function and PPF depending on the type of imposed constraints

	Parameter μ estimates	
Angle φ, degrees	Benchmark model	Concavity and null-jointness constraints
95	0.000	0.000
90	0.000	0.000
85	0.000	0.000
80	0.000	0.000
75	0.003	0.000
60	0.023	-0.005
55	0.027	-0.018
50	0.023	-0.023
45	0.016	-0.280
40	0.002	-0.325
35	-0.451	-0.220
30	-0.521	-0.182
25	-0.135	-0.883
20	-0.009	-0.146
15	-0.009	-0.021
10	0.000	0.000
5	0.000	0.000

Note: Angle φ is formed by the directional output vector with the vertical axis. Parameter μ in the directional output distance function specification is negative for the concave PPF.

constraints added to the benchmark model. We run our estimation for a series of the directional output vectors. Since we know that the underlying true PPF for this dataset is concave in b, according to Proposition 1 and the discussion in Section III, parameter μ should be estimated to be negative.

We observe that for the directional output vectors that are either too flat or too steep (i.e., for $\varphi > 75°$ and for $\varphi < 15°$) both the distance function and the PPF are estimated to be of zero curvature. The case of the most "popular", 45-degree directional output vector, is rather instructive in that if we do not impose any concavity constraints, the PPF is estimated to be convex. Imposing concavity and null-jointness constraints, the estimates are consistent with the known properties of the simulated dataset.

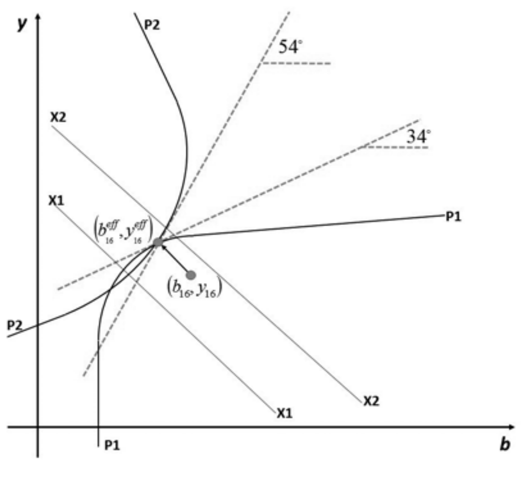
For a range of directional vectors for which $35° < \varphi < 75°$, estimating the parameters of the directional output distance function without imposing concavity and null-jointness constraints results in the wrong inference about the curvature type of the underlying PPS. Imposing these constraints, however, allows one to avoid concluding that the directional distance function is convex and its PPF is convex in b.

We illustrate in Figure 5 below what happens when concavity and null-jointness constraints are added to the estimation procedure (4). We analyze observation number 16 in the simulated dataset for the case of the "conventional" directional vector $\varphi = 45°$.

(b_{16}, y_{16}) is the combination of the values of the good and bad output in our simulated sample. The true PPF is approximated at the efficient projection $\left(b_{16}^{eff}, y_{16}^{eff}\right)$ taken along the 45° directional vector by the convex part of the parabola *P2-P2* in case the concavity and null-jointness constraints are not imposed, see Table 2. In this case the slope of the PPF is estimated to be 54°. The imposition of concavity and null-jointness constraints results in the approximating parabola's branches looking in the South-Eastern direction with the concave part of the parabola *P1-P1* approximating the true PPF at $\left(b_{16}^{eff}, y_{16}^{eff}\right)$. The PPF slope in this case is estimated to be equal to 34°. By Proposition 1, the symmetry axes *X2-X2* and *X1-X1* of the two parabolas are parallel to each other and intersect the vertical axis at the angle $\varphi = 45°$, which is the same with the directional vector angle.

In case the concavity and null-jointness constraints are not imposed, not only the curvature type of the approximating parabola may be estimated wrongly, but also the MACs are likely to be different in the two cases. Indeed, the MAC for the

Figure 5. The effects of imposing concavity and null-jointness constraints



convex parabola is estimated at 1.36, while the MAC is 0.68 when concavity and null-jointness constraints are imposed, which is also closer to the actual MAC value of 0.71 which is known because the exact shape of the true PPF is known.

B. Suggestions for the choice of directional output vectors

It is interesting to notice that in the overwhelming majority of the empirical studies applying the directional output distance function approach to the estimation of marginal abatement costs, the 45-degree directional output vector is used almost "by default." Our simulation exercise demonstrates that the choice of this and, in fact, any other vector without imposing any additional constraints may result in the wrong inference on the curvature of the PPF and the directional output distance function. We therefore suggest choosing the directional output vector based on the prior knowledge of the PPS properties, such as the type of curvature of its PPF. We would therefore reject the choice of those directional output vectors for which the directional output distance function and, consequently, the PPF, are estimated to be convex when there are strong reasons to believe that they are concave.

V. Conclusion

In this study we explored the geometric properties of quadratic approximation to the directional output distance function commonly used in empirical research on the marginal abatement costs of the undesirable outputs. The translation property imposed on the estimated parameters of the quadratic approximation ensures that the true PPF is approximated at the observed combinations of outputs either by parabolas, or by the straight lines. While economic theory implies that both the directional output distance function and the PPF for a polluting technology $y = f^{PPF}(b)$ have to be concave, the corresponding concavity constraints are usually not imposed. We demonstrate the danger of omitting such constraints by simulating a sample of observations based on a concave PPF and running a conventional estimation procedure on it to demonstrate that the true PPF that is known to be concave in b may be approximated by the convex, rather than the concave, parabolas, which is contradicting economic theory. By adding additional constraints ensuring concavity of the approximating second-order curve and a necessary condition for the null-jointness property, we observe a significant change in both the parameters of the estimated approximation to the directional output distance function, and the marginal abatement costs.

We believe an important insight in this paper is that it is erroneous to interpret the second-order curve in b and y obtained by plugging in the observed values of production inputs into the estimated quadratic specification of the directional output distance function as an individual PPF. Indeed, the behavior of the approximating curve outside of the small vicinity around the efficient projection of the observed output mix provides little inference on the behavior of the true PPF.

Although the choice of a particular directional vector is not the main subject of this study, we would suggest rejecting those vectors for which the estimated parameters of the directional output distance function imply properties of the underlying PPS that are in conflict with one's prior knowledge about it. Thus, in our simulation example, since we know in advance that the PPF for the simulated PPS is concave in the bad output b, it would be rational not to employ vectors corresponding to $\varphi \geq 75°$, i.e., the relatively flat vectors, because for those vectors the implied PPF is estimated to be convex rather than concave.

Finally, it is worthwhile asking whether our analysis will change in the multi-output case that involves several bad outputs, which is not uncommon since polluting technologies tend to produce more than a single type of harmful emission. While the directional output distance function can be easily extended to that case with the translation property constraints already derived for the multi-output case in Fare et al. (2006), the properties of the Hessian corresponding to the multi-output directional output distance function are not obvious. For instance, it is not clear whether the translation property will ensure the Hessian to be identically equal to zero, like we show is the case of the two-output polluting technology. It is not clear either whether translation property in the multi-output case will guarantee the equivalence between concavity of the directional output distance function and that of the PPF. If this equivalence is not guaranteed, the implication for the empirical work will be to impose two groups of concavity constraints: one ensuring the concavity of directional output distance function, and the other one providing for the PPF concavity. The analysis of these issues is outside the scope of this study, being the subject of our ongoing research.

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