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MARKET RISK AND VOLATILITY IN THE BRAZILIAN STOCK MARKET

JOE AKIRA YOSHINO*

Universidade de São Paulo

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We estimate in this paper the market risk implied by the prices of different options traded in the Brazilian stock market. The fundamental theory to handle this problem is the one implied by the Arrow-Debreu contingent claim concept. Using that theory, we are able to construct the term structure of market risk, and to obtain a surface that provides slices for a particular "volatility smile." The methodology that we use follows the one proposed by Shimko (1993), which is able to calculate a non-lognormal probability density function (PDF) consistent with the volatility observed in a relatively small sample of option prices. This methodology goes beyond the one proposed originally by Black and Scholes (1973), since it does not require log-normality of the PDF nor that volatility remains constant.

JEL classification codes: G12, G13

Key words: Arrow-Debreu contingent claim, options, Black-Scholes, market risk, volatility, Brazilian stock market

I. Introduction

The objective of this paper is to assess the market risk implied by the volatility of the prices of the options traded in the Brazilian stock market. In order to do that, we try different approaches based on an estimation of the

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* Professor of Economics and Finance, USP (Universidade de São Paulo), FEA (Faculdade de Economia, Administração e Contabilidade), Department of Economics, USP Professional M.Sc. Program: Mathematical Modeling in Finance www.ime.usp.br/~mpmmf. Av. Prof. Luciano Gualberto, 908 5508-900 São Paulo, Brasil. E-mail: pyoshino@usp.br. We thank for the critics, suggestions, comments and contributions that were made by David Orosco, José A. Scheinkman, one anonymous referee, and the participants of the XXI Brazilian Econometrics Society Meeting.

implied risk-neutral non-lognormal probability density function (PDF) for the underlying assets.

A fundamental theory for estimating the implied PDF is the one assumed by the Arrow-Debreu (A-D) contingent claim concept. This concept departs from the idea that there exists a primitive security that pays a unit of consumption if a specific state of nature occurs, and nothing else in any other state. Those states are represented as nodes of a binomial tree and, as we go along that tree, we need to estimate the stochastic stock return and a non-constant volatility in each node.

The A-D contingent claim theory is a general equilibrium approach, and it does not imply the existence of a global constant volatility. It therefore differs in those two aspects from the standard model of option pricing originally proposed by Black and Scholes (1973), which is a partial equilibrium model with constant volatility. In order to calibrate a model that exhibits non-constant volatility, it is therefore required to use a different methodology, such as the one proposed by Shimko (1993). That is what we do in this paper, using data from the Brazilian stock market.

The paper is organized as follows. Section II provides the basic concepts for estimating the risk-neutral PDF by using the A-D contingent claim concept. Section III describes the data and the period used to estimate the risk-neutral PDF, while section IV shows the empirical results obtained. Finally, section V presents the main conclusions.

II. Theoretical Issues

Suppose that N individuals (i = 1, 2...N) have the same beliefs about the probability of occurrence of a certain state w: $\{\Pi_w; w \in \Omega\}$. Assume further that their utility function is time-additive and separable across states:

$$U_{iw}(c_{io}, c_{iw}) = U_{io}(c_{io}) + U_{i1}(c_{iw}) \tag{1}$$

where $U_{i1}(c_{iw})$ is the utility level of the ith individual in period 1, derived from consuming the state contingent commodity c_{iw}, and c_{io} is the present consumption of the ith individual.

The maximization of this utility function subject to the individual budget constraint results in the following Lucas'(1978) asset-pricing formula:

$$S_j = \sum_{w \in \Omega} \frac{\Pi_w \ u_1(C_W)}{u_0(C_0)} \ \tilde{\chi} \ = \ E[\frac{u_1(C_W)}{u_0(C_0)} \tilde{\chi}] \ = \ \sum_{w \in \Omega} \phi_w x_{jw} \tag{2}$$

where,

$$\phi_w = \frac{\Pi_w \ u_1(C_W)}{u_0(C_0)} \tag{3}$$

Equation (2) shows that the price of a complex asset j is equal to the covariance between the expected value of the asset pay-off and the marginal rate of substitution between present and future consumption. A portfolio of elementary A-D contingent claims can replicate such a complex asset. In this sense, the price of asset S_j is its future pay-off for every possible state of nature x_{jw}, multiplied by its relative price ϕ_w (i.e., the price of a state contingent claim that pays one unit of consumption if the state w occurs).

Equation (2) is the most important formula in finance for the purpose of pricing any complex security. This formula can be specialized to obtain the Modigliani-Miller theorem, the standard CAPM, C-CAPM, I-CAPM models, and the Black-Scholes formula (see Huang and Litzenberger, 1988, and Merton, 1990). Another use of this formula is to address the equity premium puzzle (see Mehra and Prescott, 1985, and Kocherlakota, 1996).

There are three main approaches in the literature for estimating the A-D state-prices or their equivalent PDF. In the first approach, the Black-Scholes and Merton models are used. In the second approach, a PDF with a parametric form is assumed (as in Jarrow and Rudd, 1982; Longstaff, 1992; Shimko, 1993; Derman and Kani, 1994; and Madan and Milne, 1994). In the third approach, the PDF is estimated non-parametrically (this line of research is due to Aït-Sahalia, 1996, and Aït-Sahalia and Lo, 1998).

We have already mentioned that the Black-Scholes methodology uses a partial equilibrium model with constant volatility. This is because Black and Scholes (1973) assume that the stock returns follow a standard geometric Brownian motion process, and that volatility is independent from both time and prices of the underlying object. As the risk-free interest rate is also invariant, the results obtained are a lognormal PDF and a closed form solution for the stochastic differential equation that describes the evolution of the underlying object.

Given that the observed volatility in most markets is not constant through time and with the price of asset, in practice we need to infer the implied volatility by using its option price. This generates a volatility surface that changes with both maturity and moneyness, and allows us to go beyond the Black-Scholes world. In order to do that, we need to develop a method to estimate the implied PDF and the price of an A-D contingent claim. This can be done by making a portfolio of European call options, which forms a so-called butterfly spread (see Banz and Miller, 1978; Breeden and Litzenbeger, 1978, and Ross, 1976).

The first step for estimating the price of an A-D contingent claim is to use the Black-Scholes model to represent the price of a European call option on stock. This is:

$$c(t,x,T) = e^{-r\tau}\int_{x}^{\infty}(S_T - x)\ q(S_T)\ dS_T \tag{4}$$

The second derivative of this option price with respect to the exercise price renders the present value of the risk neutral PDF. Therefore:

$$\frac{\partial^2 C}{\partial x^2} = e^{-r\tau}q(S_T) = \Phi_w \tag{5}$$

Given sufficient prices of options in the same asset and with the same maturity, but with different exercise prices (X), we can obtain $\phi_w(X, t, T)$ (the price of elementary A-D contingent claim), which will compose the discounted implied PDF. This claim pays one unit of consumption only if $S_T > X$, and its price gives the probability that the price of an asset can be X. A portfolio containing the whole range of state contingent claims whose value will be \$1 is therefore equivalent to a risk-neutral debt.

Each A-D contingent claim can be replicated by a portfolio of European call options that forms a butterfly spread whose price is equal to the price of this elementary claim. Thus, we have:

$$\Phi(X,t,T) = \frac{[c(t,X-\Delta X,T) - c(t,X,T)]}{\Delta \mathrm{X}} \tag{6}$$

$$- \frac{[c(t,X,T) - c(t,X+\Delta X,T)]}{\Delta \mathrm{X}}$$

Each elementary claim gives the area of the corresponding histogram for the implied PDF. Therefore, the height of each histogram is $\frac{\Phi(X,t,T)}{\Delta X}$. Taking the limit, we get the following implied PDF:

$$\lim_{\Delta X \to 0} \frac{\Phi(X,t,T)}{\Delta X} = \frac{\partial^2 C(t,X,T)}{\partial x^2} = e^{-r\tau} q(S_T) = \Phi_w \tag{7}$$

If we now generalize the Black-Scholes model, we can obtain the following expression:

$$\frac{dS}{S} = \mu \ dt + \sigma(t,S) \ dW \tag{8}$$

where σ *(t, S)* is the local volatility at each node of the implied binomial tree. This volatility is no longer constant (as it is assumed by the Black-Scholes model) but it changes with time and with the level of stock prices.

The use of a non-constant volatility in the estimation of a risk-neutral PDF has several implementation problems, which have to do with the need to handle the higher moments of the distribution, such as the excess kurtosis, the skewness and the volatility clustering. In order to do that, the literature on financial economics has proposed several methods, such as the use of Poisson jumps (Merton, 1976), stochastic volatility (Hull and White, 1987; Melino and Turnbull, 1990, and Wiggins, 1987), generalized autoregressive conditional heteroskedastic (GARCH) processes, stochastic-volatility jump-diffusion (SVJD) processes, and non-parametric estimations of the PDF such as the so-called Kernel method (Härdle, 1990).

Most of these methods require the use of very large samples, which are usually not available when we are dealing with an emerging market with a changing regime. This is particularly the case for the non-parametric approach and the SVJD diffusion process. If we have to work with a relatively small sample, the two basic methods to obtain the PDF are the one proposed by Derman and Kani (1994), and the one developed by Shimko (1993). The first of them works with a binomial tree and introduces an observed option price in each node to determine the path for the following period. This path is characterized by the transition probability (local volatility) between consecutive nodes, and it is inferred numerically by fitting the observed option prices to the smile curve.[1]

The method developed by Shimko (1993) also works with a binomial tree, but it interpolates the implied volatility domain versus moneyness to obtain a risk-neutral PDF. In the next step, the Black-Scholes formula is used to invert the interpolated smile curve, and the European call price is solved as a continuous function of the strike prices. In this stage, the estimated call pricing function can be differentiated twice to obtain the implied risk-neutral PDF. Finally, to estimate the absent tails of the risk-neutral PDF, we need to match a lognormal distribution.[2]

In this paper, we have chosen to use Shimko's method to estimate the risk-neutral PDF.[3] In order to do that we use equation (7), and we make an interpolation of the underlying call option pricing formula. This task is performed parametrically, by imposing a particular functional form for the observed call prices and by estimating its parameters in a nonlinear least square regression.

III. Description of the Data

The main sources of the data that we use in this paper are the Brazilian Commodities and Futures Exchange of São Paulo (BM&F) and the Brazilian stock index of São Paulo (IBOVESPA). Figure 1 illustrates some characteristics of the BM&F as foreign exchange risk (FX) and total traded volume. On the other hand, Figure 2 shows the two main characteristics of the Brazilian future market on IBOVESPA, which are the IBOVESPA traded volume and the stock market risk.

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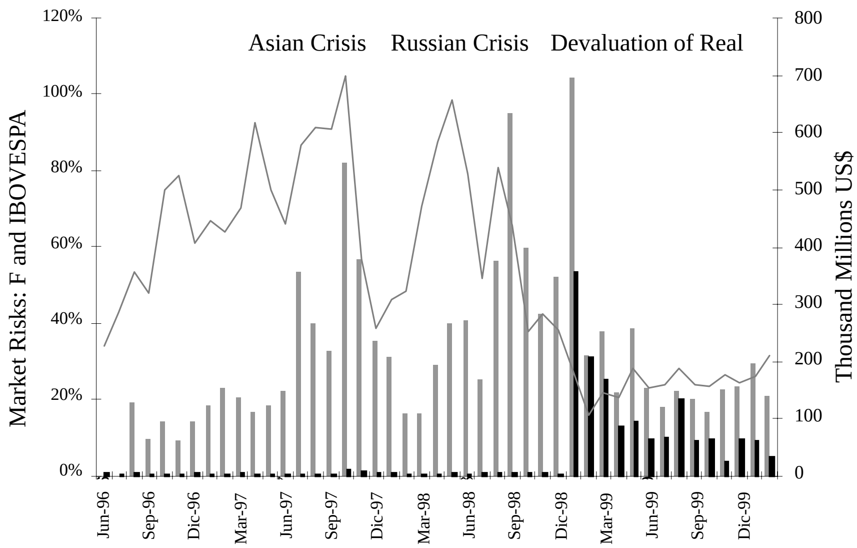
[1] Rubinstein (1994) and Dupire (1994) use a similar approach, based on the theoretical model originally developed by Cox, Ross and Rubinstein (1979).

[2] The drawback of this procedure is that it assigns a constant volatility in a critical region (the area under the left tail), which is important to measure the market risk.

[3] We have also tried the alternative Derman-Kani's method, but we did not get satisfactory results in terms of calibrating the binomial tree. That is why we have ended up using only Shimko's method.

Figure 1. Behavior of BM&F

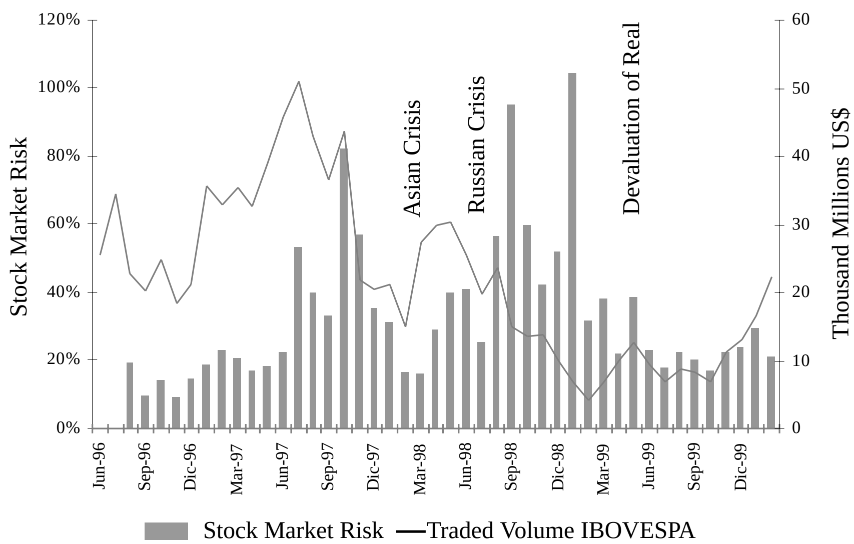


Notes: The volume traded at BM&F are composed mainly by Ibovespa Future, Daily Interbank CDI Future, Dollar Future. Historical Volatility of BM&F by using EWMA (Exponential Weighted Moving Average).

If we look at these two figures, we can notice a huge jump in the risk of the Brazilian stock market in the period before the Asian crisis of 1997, and a comparable increase before the Russian crisis of 1998. These increases can be considered as anomalous phenomena, and we will avoid them in our estimation of the market risk PDF. In Figure 2, we can also see a trade-off between stock market risk and volume traded. This is particularly clear in the Asian and Russian crises, and in the huge devaluation of the Brazilian currency (Real) that took place in 1999.

Another important element that we will take into account is the composition of the portfolio traded in the futures market. This is represented in Figure 3, which shows the breakdown of volume traded at BM&F by type of asset. We can see that, before the devaluation of the Real, the market was making an enormous speculative attack against the fixed-exchange regime (in terms of volume transacted in the dollar future).

Figure 2. Main Characteristics of IBOVESPA Future



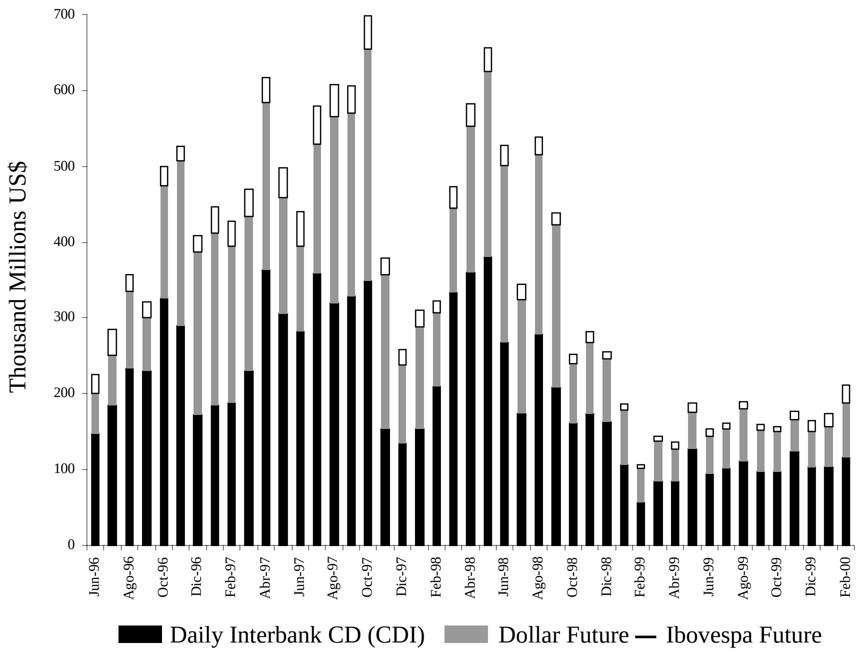
Note: Historical Volatility of IBOVESPA by using EWMA (Exponential Weighted Moving Average).

Another future contract that plays a crucial role in the Brazilian market is the CDI (One-Day Interbank Deposit Futures), whose underlying asset is the interest rate of interbank deposits. We have used this contract to obtain daily registers that allow us to estimate the yield curve for a zero-coupon interest rate. By obtaining this curve, we could calculate the term structure of interest rates, which in turn was used to discount the corresponding cash flow along a binomial tree.

To proceed with our estimation, we have selected the periods referred here as P1 (from June 1, 1996 to February 18, 1997) and P2 (February 19, 1997 to July 30, 1997), which cover the two years immediately before the Asian crisis. We have chosen those periods because they provide the longest recent series not affected by anomalous crashes, and because, by using them, our results can be compared to the ones obtained by Campa, Chang and Refalo (1999). It is worth noting, however, that these authors estimated the risk-neutral PDF for the Brazilian Real against US dollar, while we are dealing with the risk implied by the Brazilian stock market.

Given our small available sample, we have assumed a geometric Brownian

Figure 3. Historical Composition of the Volume Traded at BM&F



Note: These products represent 90% of the volume traded at BM&F.

motion process with changing volatility. This may miss some important features of the data, but it puts us in a better position to handle the higher moments of the PDF, such as the excess kurtosis (fat tailed distributions), the skewness, and the volatility clustering.

IV. Empirical Results

The main empirical results that we have obtained for our estimations of the volatility in the Brazilian stock market are summarized in Tables 1 and 2. Both Tables show the implied volatilities estimated using Shimko's method, for different maturity ranges.[4] Table 1 was constructed using the information

[4] In fact, it is not possible to analyze the periods of crashes by using the available data, since in those periods there were no transactions at all in the options and futures markets. What we can do, however, is to see if the implied market risk level before those crashes can be interpreted as an early warning of a crash.

obtained in the market of ATM (at the money) call options, while Table 2 was constructed using the information obtained in the market of ITM (in the money) call options. The other possible type of call options, called OTM (out of the money) options, was not used here, since its market was non-existent in most of the periods analyzed.

As we see in the Tables, the mean and the standard deviation of the volatilities follow different regimes for the selected periods and the different

Table 1. Implied Volatility. Mean and Standard Deviation for Brazilian Stock Market. ATM Option

Maturity range	Period before Asian crisis: P1: 06/01/96 - 02/18/97	Period before Asian crisis: P2: 02/19/97 - 07/30/97
7-15 days		
Mean	20.02%	23.64%
Standard deviation	5.52%	4.85%
# observations	30	31
16-21 days		
Mean	22.18%	21.52%
Standard deviation	8.60%	5.18%
# observations	40	22
22-30 days		
Mean	22.33%	25.80%
Standard deviation	4.09%	4.66%
# observations	46	40
31-45 days		
Mean	23.47%	26.04%
Standard deviation	3.86%	3.68%
# observations	69	67
46-60 days		
Mean	24.63%	26.19%
Standard deviation	5.32%	4.20%
# observations	93	72

Table 2. Implied Volatility. Mean and Standard Deviation for Brazilian Stock Market. ITM Option

Maturity range	Period before Asian crisis: P1: 01/06/96 -02/18/97	Period before Asian crisis: P2: 02/19/97 -07/30/97
7-15 days		
Mean	56.30%	61.83%
Standard deviation	55.17%	46.72%
# observations	5	11
16-21 days		
Mean	56.89%	43.29%
Standard deviation	39.54%	24.31%
# observations	7	11
22-30 days		
Mean	25.14%	39.59%
Standard deviation	7.57%	27.00%
# observations	17	31
31-45 days		
Mean	29.92%	38.32%
Standard deviation	18.74%	24.61%
# observations	22	54
46-60 days		
Mean	32.11%	53.68%
Standard deviation	19.13%	30.10%
# observations	23	34

maturity ranges. For example, the implied volatility tends to increase with maturity for the ATM options but it does not behave in the same way in the ITM option market. This result is similar to the one obtained by Aït-Sahalia, Wang and Yared (2001) for the US option market, and it can be interpreted as a signal of an imminent crash. What it shows is that the different options are in different portions of a certain volatility surface, which has a larger mean and standard deviation in the ITM region than in the ATM region. This could also be seen when we compare Tables 1 and 2, for both period P1 and period P2. In his study of the Brazilian foreign exchange risk market, Campa, Chang and Refalo (1999) also interpret the behavior of the volatility as a signal of an imminent crisis. Their results, however, are in a sense misleading, since they do not disentangle the different implied volatilities in terms of the ATM and ITM call options, and put them together without considering the fact that they lie in different regions of the volatility surface.

A third result that we have obtained after estimating the implied volatilities is that the market risk calculated using historical volatility results underestimates the true measure of risk in periods before a crisis. This can be seen on Figure 4, which shows the implied market risk of an ATM call option using Shimko's method versus the historical volatility calculated using the more traditional RiskMetrics methodology.

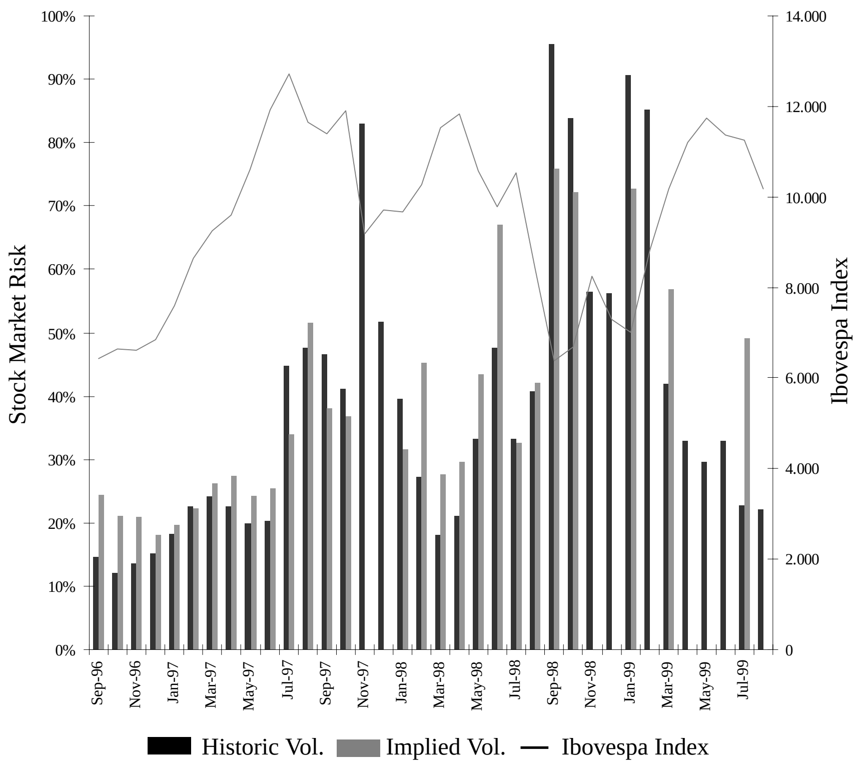
As was stated above, another difference between the implied volatility estimated by Shimko's method and the historical volatility estimated using more traditional methods is that its PDF is not necessarily lognormal. In Figure 5, for example, we have depicted the implied Shimko's PDF against the corresponding lognormal distribution that has the same mean and variance.[5] Comparing both lines we see that the implied Shimko's PDF has a negative skewness and a more leptokurtic behavior than the lognormal PDF. This can be interpreted as a signal that the Brazilian stock market shows a pattern of non-log-normality, which suggests that it is beyond the Black-Scholes world.[6]

One important consequence of non-log-normality in the Brazilian case is its particular implication in terms of market risk, due to the fact that the left tail of the PDF is considerably more dense than the corresponding one under the assumption of log-normality. For example, if we want to estimate the minimum value of the IBOVESPA at a certain moment in the future with a certain level of confidence, we will get a higher number using a lognormal PDF than the one that we will obtain using Shimko's PDF. This implies that the assumption of log-normality, in this particular case, underestimates market

[5] In fact, the estimated implied PDF has kinks that are inherent to Shimko's method. They have been adjusted using a lognormal distribution for the two missing tails.

[6] These stylized facts have also been reported by Aït-Sahalia and Lo (1998) for the US stock market.

Figure 4. The Implied Versus Historical Volatility Brazilian Stock Market

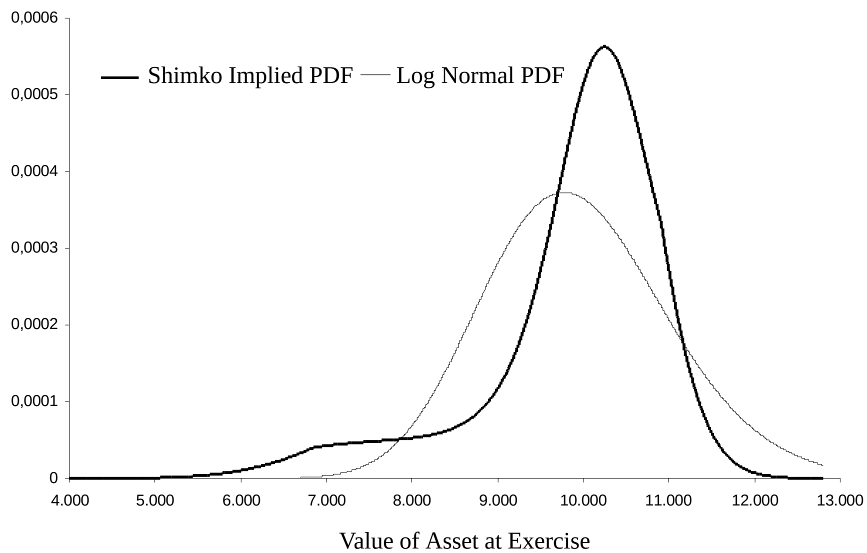


risk even if we use the same estimates for the mean and the variance of market returns.[7]

The implied volatility PDF has also great value for two additional reasons. First, it can be used for inferring the diffusion process of the underlying asset. Second, for purposes of risk management, it provides a measure of the so-called economic-VaR, rather than the statistical VaR calculated using

[7] Let us assume, for example, that the IBOVESPA takes a value of 13,000. If we want to determine the minimum value for that index in the next 30 days with 95% of confidence, we obtain a value of 11,729 using a lognormal PDF, and a value of 11,402 using the implied Shimko's PDF. By using the first of those methods, therefore, we underestimate the stock market risk exposure by 2.8%.

Figure 5. The Implied Shimko PDF and Log Normal PDF: Just before Asian Crisis. February - July, 1997



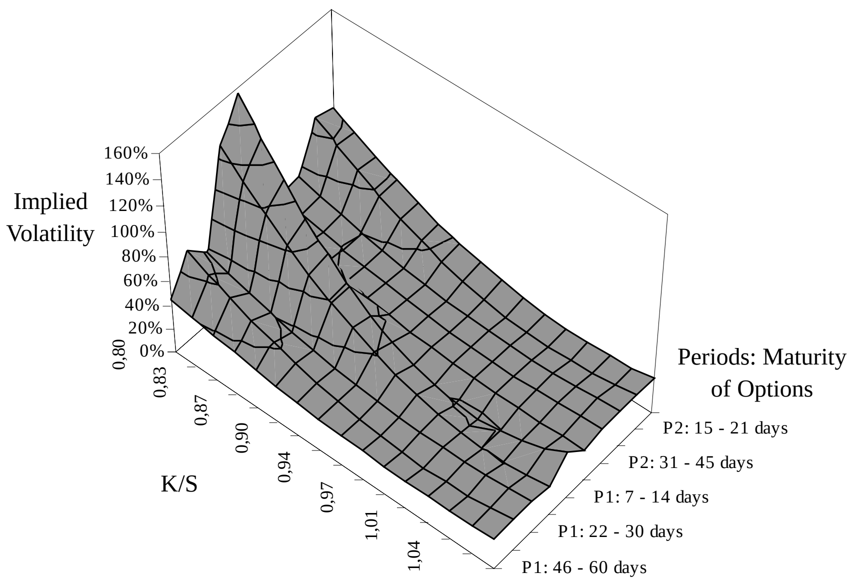
RiskMetrics or other similar methodologies.[8] Actually, we can say that the implied PDF represents the economic evaluation of the market risk expected by the agents.

We have already seen on Tables 1 and 2 that, in the Brazilian market, ITM call options have larger implied volatilities than ATM call options. We have also said that this has to do with the idea that the different options occupy different portions of a certain volatility surface, which is steeper for the ITM options, flatter for the ATM options, and even flatter for the OTM options. This idea can be seen more clearly if look at Figure 6, which shows the volatility surface or the so-called term structure of market risk for periods P1 and P2. The different portions of the surface are related to the different possible ranges of moneyness, measured as the ratio between the strike and the spot prices of an asset (K/S). This ratio measures the intrinsic value of an option at its maturity, and we can relate the ITM region with areas where K/S < 1, the ATM region with areas where K/S = 1 and the OTM region with areas where K/S > 1.

[8] For a discussion about this, see Aït-Sahalia, Wang and Yared (2001).

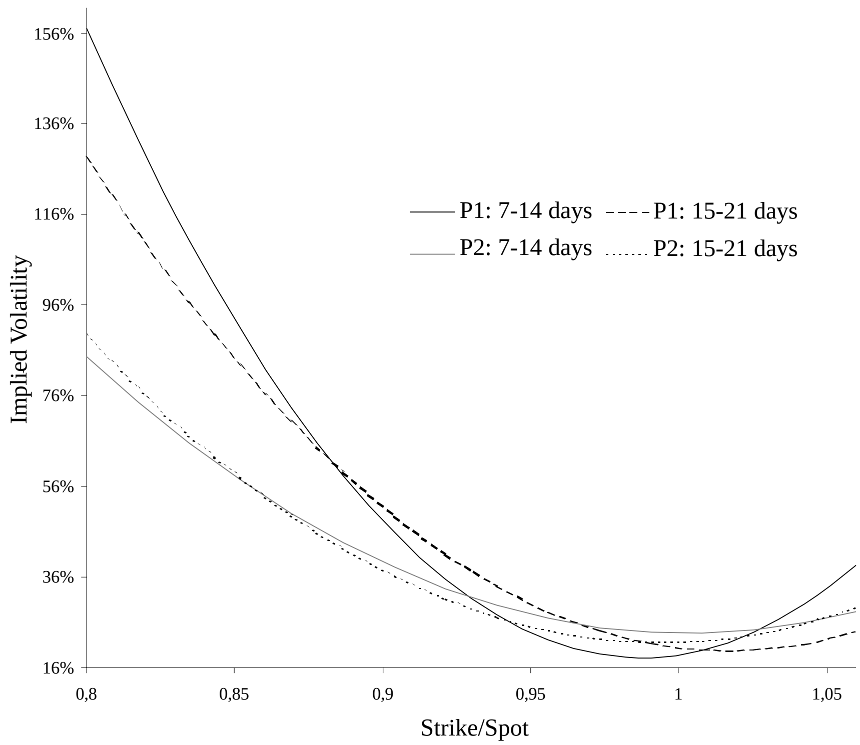
Figure 6 was obtained by calculating weighted average volatilities for each very short range of moneyness. By this procedure we got approximately 30 points for each slice in the plane of volatility versus moneyness, and we interpolated those points using a quadratic function that represents each smile curve. Some of those smile curves are depicted on Figure 7.

Figure 6. The Term Structure of Anticipated Market Risk Brazilan Stock Market: Before Asian Crisis



The main conclusion that we can obtain from Figure 7 is that short-run volatility was higher in period P1 than in period P2. This can be seen in the fact that the corresponding smile curves are steeper, especially in the ITM option region (that is, for moneyness levels such that K/S < 1). In other words, this implies that prices of the short-term ITM options were higher in period P1 than in period P2 (due to a larger probability of obtaining a capital gain) and it can be related to the behavior of the IBOVESPA that we have graphed on Figure 4. During period P1, this index was growing rapidly, and the probability of continuing growth was high. In contrast, during period P2 the index was also growing, but market risk increased sharply. This can be

Figure 7. The Volatility Smile. Periods Before the Asian Crisis



associated to a higher probability of a crash, and for this reason the volatility smile surface became flatter.

Figure 7 also shows that the implied volatility for the ATM options (that is, the ones that lie in the region where moneyness is around a level of K/S = 1) is higher in period P2 than in period P1. If we consider that the Asian crisis happened at the end of period P2, we can interpret this result saying that the ATM option market has provided better information than the ITM option market to predict that crisis. On the other hand, the ITM options have provided a better signal of the speculative thinking that takes place when there is a higher chance of growth in the stock market, and that is why they tended to have higher prices in period P1. This reasoning also gives a hint on why we observe no transactions in the OTM option market, since, when the market is anticipating a crash, there is no scenario in which the agents could make a gain by buying an option that is currently out of the money.

V. Conclusions

The concept of the Arrow-Debreu contingent claim provides a useful foundation to estimate implied stock market volatilities, especially because it does not require to assume that volatility remains constant and follows a lognormal PDF (as does the more traditional Black-Scholes methodology).

The practical application of this concept can be implemented in a number of ways, of which we have chosen Shimko's method. This method has the advantage that it does not require a very large sample of observations, and that is very efficient in terms of using the information that we can gather in the different option markets.

When we applied this method to the Brazilian case, we obtained estimations of the implied volatilities that differ for the ATM call options and the ITM call options. We found that, in the period before the Asian crisis of 1997, the implied volatility of the ATM options provides a valuable signal for the expected downside movement of the stock market, while the implied volatility of the ITM options provides useful information about the speculative thinking that takes place when market risk is acceptable.

Of course, our calibrated Shimko tree and its corresponding PDF does not necessary hold for other periods when local volatility changes, but it gives a methodology for inferring the diffusion process of the underlying assets. This tree can also be used for pricing path dependent derivatives on the same underlying asset, while its associated PDF can be a useful risk management tool, since it is able to estimate a more realistic economic-VaR than the statistical-VaR calculated with other methodologies such as RiskMetrics.

References

Aït-Sahalia, Y. (1996), "Nonparametric Pricing of Interest Rate Derivative Securities," *Econometrica* **64**: 527-560.

Aït-Sahalia, Y., and A. Lo (1998), "'Nonparametric Estimation of State-Price Densities Implicit in Financial Asset Prices," *Journal of Finance* **53**: 499-547.

Aït-Sahalia, Y. Wang, and F. Yared (2001), "Do Options Markets Correctly Price the Probabilities of Movement of the Underlying Asset?," *Journal of Econometrics* **102**: 67-110.

Banz, R., and M. Miller (1978), “Prices for State Contingent Claims: Some Estimates and Applications,” *Journal of Business* **51**: 653-672.

Black, F., and M. Scholes (1973), “The Pricing of Options and Corporate Liabilities,” *Journal of Political Economy* **81**: 637-659.

Breeden, D., and R. Litzenberger (1978), “Prices of State-Contingent Claims Implicit in Option Prices,” *Journal of Business* **51**: 621-651.

Campa, J., Chang, K., and J. Refalo (1999), “An Options-Based Analysis of Emerging Market Exchange Rate Expectations: Brazil’s Real Plan, 1994-1997,” NBER Working Paper Series, WP 6929.

Cox, J., Ross, S., and M. Rubinstein (1979), “Option Pricing: A Simplified Approach,” *Journal of Financial Economics* **7**: 229-263.

Derman, E., and I. Kani (1994), “Riding on the Smile,” *RISK* **7**: 32-39.

Dupire, B. (1994), “Pricing with a Smile”, *RISK* **7**: 18-20.

Härdle, W. (1990), *Applied Nonparametric Regression*, Cambridge, UK, Cambridge University Press.

Huang, C., and R. Litzenberger (1988), *Foundations for Financial Economics*, Amsterdam, North-Holland.

Hull, J., and A. White (1987), “The Pricing of Options on Assets with Stochastic Volatilities,” *Journal of Finance* **42**: 281-300.

Jarrow, R., and A. Rudd (1982), “Approximate Option Valuation for Arbitrary Stochastic Processes,” *Journal of Financial Economics* **10**: 347-369.

Kocherlakota N., (1996), “The Equity Premium: It’s Still a Puzzle,” *Journal of Economic Literature* **34**: 42-71.

Longstaff, F. (1992), “An Empirical Examination of the Risk-Neutral Valuation Model,” Working Paper, College of Business, Ohio State University, and the Anderson Graduate School of Management, UCLA.

Lucas, R.E. Jr. (1978). “Asset Prices in an Exchange Economy,” *Econometrica* **46**: 1429-1445.

Madan, D., and F. Milne (1994), “Contingent Claims Valued and Hedged by Pricing and Investing in a Basis,” *Mathematical Finance* **4**: 223-245.

Mehra, R., and E. Prescott, (1985), “The Equity Premium: A Puzzle,” *Journal of Monetary Economics* **15**: 145-161.

Melino, A., and S. Turnbull (1990), “Pricing Foreign Currency Options with Stochastic Volatility,” *Journal of Econometrics* **45**: 239-265.

Merton, R. (1976), "Option Pricing when Underlying Stock Returns Are Discontinuous," *Journal of Financial Economic*s **3**: 125-144.

Merton, R. (1990), *Continuous-Time Financ*e, Cambridge, Massachusetts, Blackwell.

Ross, S. (1976), "Options and Efficiency," *Quarterly Journal of Economics* **90**: 75-89.

Rubinstein, M. (1994), "Implied Binomial Trees," *Journal of Finance* **49**: 771-818.

Shimko, D. (1993), "Bound of Probability," *RISK* **6**: 33-37.

Wiggins, J. (1987), "Option Values under Stochastic Volatility: Theory and Empirical Estimates," *Journal of Financial Economics* **19**: 351-372.