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THE OPPORTUNITY COST OF BEING CONSTRAINED BY THE TYPE OF ASSET: BONDS ONLY OR STOCKS ONLY

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I explore investors' welfare losses when they restrict themselves to invest in either stocks only or bonds only, but not in both. The restriction gives investors sub-optimal asset allocations that result in welfare losses. To measure these welfare losses I compare "only stock indices and Treasury bills" optimal portfolios and "only bond indices and Treasury bills" optimal portfolios with "stock and bond indices and Treasury bills" optimal portfolios using the concept of proportionate opportunity cost along with various CRRA utility functions. The original historical asset returns data set is used with a VAR in generating joint returns distributions for the portfolio formation period. I show that for investors with low levels of risk aversion welfare losses do not exceed 1.8% of initial wealth when they invest sub-optimally. For investors with medium and high levels of relative risk aversion, sub-optimal portfolios of only one type of assets, stocks only or bonds only, along with Treasury bills, give expected utility about as high as optimal portfolios that include both types of assets, stocks and bonds.

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I. Introduction

A key question in the literature on portfolio choice is: will one-type-asset portfolios be well-diversified? In other words, will one diversify sub-optimally if one invests in a portfolio that contains stocks only, or bonds only, as opposed to a portfolio that contains both stocks and bonds?

Economists researching household portfolio choices find that asset allocation decisions are influenced by several factors including degree of risk aversion,

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education level (which affects the investor's ability to gather and process financial information), level of initial wealth, age category, borrowing constraints, family size, and gender: Bertaut (1998), Bertaut and Haliassos (1997), Guiso, Jappelli, and Terlizzese (1996), Guiso, Haliassos, and Jappelli (2002), Haliassos and Michaelides (2003), Jagannathan and Kocherlakota (1996), Jianakoplos and Bernasek (1998), Kennickell, Starr-McCluer, and Surrette (2000), Uhler and Cragg (1971), Ameriks and Zeldes (2000), Attanasio and Hoynes (2000).

Although these studies provide a comprehensive catalogue of factors that affect household portfolio choice, they do not address the impact of sub-optimal asset allocation on household-investors' utility when one or more types of assets are excluded from portfolios. Haliassos and Bertaut (1995) and Bertatut (1998), using data from the Survey of Consumer Finances, report that only 20% or less of all U.S. households hold stocks or shares of mutual stock funds in their portfolios. Both studies suggest that the primary reasons for such portfolio choices are the level of education of investors, cultural factors, the cost of obtaining and processing financial information. Haliassos and Michaelides (2003) state that "…relatively small, fixed, stock market entry costs are sufficient to deter households from participating in the stock market".

Although these factors may explain either the presence or absence of certain types of asset in household portfolios, the question that remains unanswered is: How big is one's welfare loss if one restricts oneself to invest in one type of asset only, rather than to invest in a portfolio of both bonds and stocks?

Estimating utility losses for an investor who restricts himself to invest in only one type of asset is important for analyzing portfolio allocation decisions. Investors whose level of risk aversion is high would not participate in and, therefore, not benefit from the full asset market. And the cost of constraining themselves to only one type of asset for highly risk-averse investors would be virtually zero simply because their optimal portfolio strategy is to invest in the least risky assets – clearly, non-stock assets.

For investors with medium or low levels of risk aversion the situation is different. Their levels of risk aversion will permit them to participate in and, therefore, to benefit from the full asset market. Their costs from non-participation would be significantly larger than those of highly risk-averse investors.

II. Proportionate opportunity cost

In order to measure welfare losses from investing in constrained portfolios, I will compare expected utility from the optimal portfolio constrained to include bonds only or stocks only, with that from the optimal unconstrained portfolio containing both bonds and stocks, by using the concept of opportunity cost as developed by Brennan and Torous (1999) and Tew, Reid and Witt (1991).

Proportionate opportunity cost is the best way to measure investors' welfare losses because results are readily interpretable as intuitively "large" or "small", which would not be true if compensating payments were expressed in additive dollar terms. Under the assumption of the constant relative risk aversion (CRRA) utility function:

$$U(\tilde{w}) = \begin{cases} \frac{1}{\gamma}\tilde{w}^{\gamma}, & \gamma < 1,\ \gamma \neq 0,\ \tilde{w} > 0 \\ -\infty, & \tilde{w} \leq 0 \end{cases}, \tag{1}$$

the proportionate opportunity cost (willingness to accept payment as compensation for being constrained to only one type of assets) can be calculated as θ-1 where θ is defined by

$$EU(\theta w_0 \tilde{R}_B^*) = EU(w_0 \tilde{R}_{B\&S}^*), \tag{2}$$

$$EU(\theta w_0 \tilde{R}_S^*) = EU(w_0 \tilde{R}_{B\&S}^*), \tag{3}$$

where w_0 is the initial wealth, $R_{B\&S}^*$, R_B^*, and R_S^* are the stochastic returns per dollar invested for, respectively, the optimal portfolio with both bonds and stocks (B&S), for the portfolio with bonds only (B), and for the portfolio with stocks only (S). Solving (2) and (3) with the utility function (1) gives

$$\theta = \left[\frac{E(\tilde{R}_{B\&S}^*)^{\gamma}}{E(\tilde{R}_B^*)^{\gamma}}\right]^{\frac{1}{\gamma}} \tag{4}$$

and

$$\theta = \left[\frac{E(\tilde{R}_{B\&S}^*)^{\gamma}}{E(\tilde{R}_S^*)^{\gamma}}\right]^{\frac{1}{\gamma}}. \tag{5}$$

Under the CRRA utility function θ also equals the ratio of certainty equivalents of the bonds-and-stocks unconstrained optimal portfolios, and bonds only or stocks only constrained optimal portfolios. Since the ratio of certainty equivalents is unitless, and in particular has no time units, the proportionate opportunity cost, θ-1, is also timeless. But its numerical value depends on the number of months until the horizon, i.e., with an investment horizon of T months, the proportionate willingness to accept payment to accept the constraint is θ^T.

III. The procedure

This section of the paper describes the procedure for forming investors' portfolios, for inferring the joint probability distribution function of asset returns via a vector autoregression, for computing the constrained optimal and unconstrained optimal portfolios, and for the calculation of the proportionate opportunity cost.

A. Portfolio formation

The data set used is monthly historically occurring asset returns over the ten-year period from January 1992 through December 2001. The source is Ibbotson Associates (2002). With different time units, the conclusions might be affected. But with quarterly asset returns I would need to extend the time period to 1972.I to 2001.IV, and with annual asset returns the new time period will be from 1882 to 2001 just to get the same 120 data points. In both cases I would have to deal with very old asset returns that might not accurately reflect the true probability distribution facing current investors. The choice of monthly time units is also consistent with the studies of Simaan (1993) and Kroll, Levy, and Markowitz (1984).

To form the "only stocks" constrained portfolio I use two composite stock indices: S&P 500 and NASDAQ, with Treasury bills as the nominally risk-free asset. To form the "only bonds" constrained portfolio I use two composite bond indices: Salomon Brothers' Long-Term High-Grade Corporate Bonds Index, and Long-Term Government Bonds Total Return Index, with Treasury bills as the nominally risk-free asset. The unconstrained portfolio includes both types of assets: stocks and bonds. To form the unconstrained portfolio I use the same two composite stock indices and the same two composite bond indices, and Treasury bills.

B. Vector autoregressions of returns

To get expected values and probability distributions of real returns for the four indices and Treasury bills at time $T+1$, the portfolio formation period, I estimate a vector autoregressive process (VAR). Then I derive the joint probability distribution for the four indices and Treasury bills real returns. Finally, I construct optimal constrained and optimal unconstrained portfolios.

To derive the joint probability distribution of empirical deviations from the VAR-estimated conditional means for those four indices returns and inflation, the following methodology is applied: The nominal return on index i at time t minus the nominal return on Treasury bills at time t gives the excess return on index i at time t ($x_{i,t}$) for $i=1,\dots,4$ and for $t=1,\dots,T$. Running a VAR for excess returns of those four indices and realized inflation, as

$$\begin{bmatrix} x_{1,t} \\ . \\ x_{4,t} \\ \pi_t \end{bmatrix} = \begin{bmatrix} c_1 \\ . \\ c_4 \\ c_5 \end{bmatrix} + \begin{bmatrix} v_{1,1}(L) & . & . & v_{1,5}(L) \\ . & . & . & . \\ v_{4,1}(L) & . & . & v_{4,5}(L) \\ v_{5,1}(L) & . & . & v_{5,5}(L) \end{bmatrix} \begin{bmatrix} x_{1,t} \\ . \\ x_{4,t} \\ \pi_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ . \\ \varepsilon_{4,t} \\ \varepsilon_{\pi,t} \end{bmatrix}, \tag{6}$$

$\{\hat{c}_i\}$, $\{\hat{\varepsilon}_{i,t}\}$ and $\{\hat{v}_{i,k}(L)\}$ are obtained, where

$$\hat{v}_{i,k}(L) = \hat{\delta}^1_{i,k} L^1 + \hat{\delta}^2_{i,k} L^2 + \dots \tag{7}$$

The vector of conditional expected values of excess returns for time $T+1$ and expected inflation for time $T+1$ is computed as:

$$\begin{bmatrix} E_T x_{1,T+1} \\ . \\ E_T x_{4,T+1} \\ E_T \pi_{T+1} \end{bmatrix} = \begin{bmatrix} \hat{c}_1 \\ . \\ \hat{c}_4 \\ \hat{c}_5 \end{bmatrix} + \begin{bmatrix} \hat{v}_{1,1}(L) & . & . & \hat{v}_{1,5}(L) \\ . & . & . & . \\ \hat{v}_{4,1}(L) & . & . & \hat{v}_{4,5}(L) \\ \hat{v}_{5,1}(L) & . & . & \hat{v}_{5,5}(L) \end{bmatrix} \begin{bmatrix} x_{1,T+1} \\ . \\ x_{4,T+1} \\ \pi_{T+1} \end{bmatrix}. \tag{8}$$

The expected real return on index i in period $T+1$, the portfolio formation period, is

$$\begin{bmatrix} E_T r_{1,T+1} \\ . \\ E_T r_{4,T+1} \end{bmatrix} = \begin{bmatrix} E_T x_{1,T+1} \\ . \\ E_T x_{4,T+1} \end{bmatrix} + \begin{bmatrix} r^n_{TB,T+1} \\ . \\ r^n_{TB,T+1} \end{bmatrix} - \begin{bmatrix} E_T \pi_{T+1} \\ . \\ E_T \pi_{T+1} \end{bmatrix}, \tag{9}$$

where $r^n_{TB,T+1}$ is the ex ante observed nominal return on Treasury bills for time T+1. The expected real return on Treasury bills for time T+1 is

$$E_T r_{TB,T+1} = r^n_{TB,T+1} - E_T \pi_{T+1}. \tag{10}$$

Finally, the conditional probability distribution for real returns for time T+1 is determined by

$$\begin{bmatrix} \tilde{r}_{1,T+1} \\ \cdot \\ \tilde{r}_{4,T+1} \\ \tilde{r}_{TB,T+1} \end{bmatrix} = \begin{bmatrix} E_T x_{1,T+1} \\ \cdot \\ E_T x_{4,T+1} \\ 0 \end{bmatrix} + \begin{bmatrix} r^n_{TB,T+1} \\ \cdot \\ r^n_{TB,T+1} \\ r^n_{TB,T+1} \end{bmatrix} - \begin{bmatrix} E_T \pi_{T+1} \\ \cdot \\ E_T \pi_{T+1} \\ E_T \pi_{T+1} \end{bmatrix} + \begin{bmatrix} \tilde{\varepsilon}_{1,T+1} \\ \cdot \\ \tilde{\varepsilon}_{4,T+1} \\ 0 \end{bmatrix} - \begin{bmatrix} \tilde{\varepsilon}_{\pi,T+1} \\ \cdot \\ \tilde{\varepsilon}_{\pi,T+1} \\ \tilde{\varepsilon}_{\pi,T+1} \end{bmatrix} \tag{11}$$

where $\begin{bmatrix} \tilde{\varepsilon}_{1,T+1} \\ \cdot \\ \tilde{\varepsilon}_{4,T+1} \\ \tilde{\varepsilon}_{\pi,T+1} \end{bmatrix}$ takes on the historically observed values $\begin{bmatrix} \tilde{\varepsilon}_{1,t} \\ \cdot \\ \tilde{\varepsilon}_{4,t} \\ \tilde{\varepsilon}_{\pi,t} \end{bmatrix}$ from regression (6), t=1,2,…,T, with equal probabilities (1/T).

This method for deriving asset returns probability distribution functions, using historically occurring innovations to asset returns captured through a VAR procedure, is superior to the VAR method mentioned in the earlier literature, e.g., Campbell and Viceira (2002). The earlier literature on derivation of asset returns probability distribution functions assumes that the distribution of asset returns is static, not evolving over time. But the reality is such that the asset returns distribution is dynamic, depending on both recent realizations and the fixed historical distribution of shocks to the dynamic asset returns process. Thus a better way to derive asset returns probability distribution functions is to include the dynamics of the past history of asset returns.

The probability distribution of returns derived as shown in (6)-(11) is used for both types of portfolios, "only bonds" or "only stocks", as well as for unconstrained portfolios of both bonds and stocks.

C. Constrained portfolios

Using the information about the four indices and Treasury bills' derived probability distribution of real returns, compute constrained optimal portfolios with (a) the two stock indices and Treasury bills, and (b) with the two bond indices and Treasury bills, as solutions to the problems:

$$\underset{\{\alpha_1,\alpha_2\}}{Max}\, EU(\tilde{w}) = Max\;\; E\left\{\frac{1}{\gamma}\left[w_0\left(\alpha_1\tilde{r}_{S\&P500} + \alpha_2\tilde{r}_{NASDAQ} + (1-\alpha_1-\alpha_2)\;\tilde{r}_{TB}\right)\right]^{\gamma}\right\}, \quad (12)$$

$$\underset{\{\beta_1,\beta_2\}}{Max}\, EU(\tilde{w}) = Max\;\; E\left\{\frac{1}{\gamma}\left[w_0\left(\beta_1\tilde{r}_{CorpBonds} + \beta_2\tilde{r}_{GovBonds} + (1-\beta_1-\beta_2)\;\tilde{r}_{TB}\right)\right]^{\gamma}\right\}, (13)$$

where $\alpha_1,\ \alpha_2$, 1 - α_1 - α_2, and $\beta_1,\ \beta_2, 1-\ \beta_1-\beta_2$ are the individual portfolio shares in the constrained optimal portfolios with stocks only and bonds only. To get these portfolios, search over the ($\alpha_1,\ \alpha_2$) and ($\beta_1,\ \beta_2$) spaces to maximize expected utility, using nonlinear optimization by a quasi-Newton method based on iterative solutions of the first-order maximization conditions of problems (12) and (13). Expectations are taken over the joint probability distributions derived from the 5-asset VAR.

D. Unconstrained portfolios

The next step is to obtain the unconstrained optimal portfolio with four indices and Treasury bills as the solution to the problem:

$$\underset{\{\alpha_1,\ldots,\alpha_4\}}{Max}\, EU(\tilde{w}) =$$

$$= Max\;\; E\left\{\frac{1}{\gamma}\left[w_0\left(\kappa_1\tilde{r}_{S\&P500} + \kappa_2\tilde{r}_{NASDAQ} + \kappa_3\tilde{r}_{CorpBonds} + \kappa_4\tilde{r}_{GovBonds}\right.\right.\right.$$

$$\left.\left.\left. + (1-\kappa_1-\ldots-\kappa_4)\;\tilde{r}_{TB}\right)\right]^{\gamma}\right\} \quad (14)$$

where κ_1, ..., κ_4 are the four individual indices' portfolio shares in the unconstrained optimal portfolio. To get the portfolio, search over the (κ_1, ..., κ_4) space to optimize expected utility, again using nonlinear optimization by a quasi-Newton method based on using iterative solutions of the first-order conditions of problem (14). The expectation is taken over the joint probability distribution derived from the 5-asset VAR.

E. Calculating opportunity cost

On the basis of the constrained and unconstrained optimal portfolios obtained above, we now calculate the proportionate opportunity cost, θ-1. We use the following notation: $E(\tilde{R}^*)^\gamma$ (where $\tilde{R}$ is the gross return for the optimal unconstrained portfolio with four indices and Treasury bills), and $E(\tilde{R}_B{}^*)^\gamma$ and $E(\tilde{R}_S{}^*)^\gamma$ (where $\tilde{R}_B$ and $\tilde{R}_S$ are the gross returns for the optimal constrained portfolios with either two bond indices and Treasury bills or with two stock indices and Treasury bills). $E(\tilde{R}^*)^\gamma$ is determined as follows:

$$\mathrm{E}(\tilde{R}^*_{B\&S})^\gamma =$$

$$= \frac{1}{T}\sum_{t=1}^{T}\left\{\begin{bmatrix}\kappa_1^* & \kappa_2^* & \kappa_3^* & \kappa_4^* & \kappa_5^*\end{bmatrix}\begin{bmatrix} E_T r_{S\&P500,T+1} + \varepsilon_{S\&P500,t} - \varepsilon_{\pi,t} \\ E_T r_{NASDAQ,T+1} + \varepsilon_{NASDAQ,t} - \varepsilon_{\pi,t} \\ E_T r_{CorpBonds,T+1} + \varepsilon_{CorpBonds,t} - \varepsilon_{\pi,t} \\ E_T r_{GovBonds,T+1} + \varepsilon_{GovBonds,t} - \varepsilon_{\pi,t} \\ E_T r_{TB,T+1} + 0 - \varepsilon_{\pi,t}\end{bmatrix}\right\}^\gamma, \quad (15)$$

where vector κ_i^* is the vector of optimal shares for the unconstrained portfolio (with $\kappa_5 = 1 - \kappa_1 - ... - \kappa_4$), and the vectors of $E_T r_{i,T+1} + \varepsilon_{i,t} - \varepsilon_{\pi,t}$ and $E_T r_{TB,T+1} + 0 - \varepsilon_{\pi,t}$ for t=1, ..., T are the vectors of particular possible values of real returns (conditional on the data set for times t=1 through T) at time T+1.

The constrained values are determined as follows:

$$E(\tilde{R}_S^*)^\gamma = \frac{1}{T}\sum_{t=1}^{T}\left\{\begin{bmatrix}\alpha_1^* & \alpha_2^* & \alpha_3^*\end{bmatrix}\begin{bmatrix} E_T r_{S\&P500,T+1} + \varepsilon_{S\&P500,t} - \varepsilon_{\pi,t} \\ E_T r_{NASDAQ,T+1} + \varepsilon_{NASDAQ,t} - \varepsilon_{\pi,t} \\ E_T r_{TB,T+1} + 0 - \varepsilon_{\pi,t}\end{bmatrix}\right\}^\gamma, \quad (16)$$

$$E(\tilde{R}_B^*)^\gamma = \frac{1}{T}\sum_{t=1}^{T}\left\{\begin{bmatrix}\beta_1^* & \beta_2^* & \beta_3^*\end{bmatrix}\begin{bmatrix} E_T r_{CorpBonds,T+1} + \varepsilon_{CorpBonds,t} - \varepsilon_{\pi,t} \\ E_T r_{GovBonds,T+1} + \varepsilon_{GovBonds,t} - \varepsilon_{\pi,t} \\ E_T r_{TB,T+1} + 0 - \varepsilon_{\pi,t}\end{bmatrix}\right\}^\gamma, \quad (17)$$

where α_i^* and β_i^* are the unit-sum vectors of constrained portfolio shares for the "only stocks" portfolio and for the "only bonds" portfolio.

Finally, find numerical values for θ:

$$\theta = \left[\frac{E(\tilde{R}^*_{B\&S})^\gamma}{E(\tilde{R}^*_B)^\gamma} \right]^{\frac{1}{\gamma}}, \tag{18}$$

$$\theta = \left[\frac{E(\tilde{R}^*_{B\&S})^\gamma}{E(\tilde{R}^*_S)^\gamma} \right]^{\frac{1}{\gamma}}. \tag{19}$$

The denominator expectations are taken over the distributions implied by the 5-variable VAR. The proportionate opportunity cost, θ-1, is calculated for both the "only stocks" restriction and the "only bonds" restriction. The above exercise was done for each of 11 alternative values of the risk aversion parameter, γ.

IV. Results

A. Opportunity costs

Table 1 reports the results from calculating the proportionate opportunity cost for 11 different values of relative risk aversion for two types of constrained portfolios: portfolios that consist of two stock indices and Treasury bills, the "only stocks" portfolios, and portfolios that consist of two bond indices and Treasury bills, the "only bonds" portfolios. The 11 different values of relative risk aversion include extremely low levels (from 0.7 to 3), medium levels (from 9 to 12), and extremely high levels (from 29 to 31). Even though degrees of risk aversion in excess of 10 are regarded as highly unreasonable (Mehra and Prescott 1985), it is not unthinkable that some investors might be characterized by such extreme levels of risk aversion. I use these extreme values (very low as well as very high) to illustrate the limits of the constrained investment strategies.

Of all the values of relative risk aversion examined, the lowest proportionate opportunity cost of investing in "only stocks" portfolios, 0.0% (0.000), for example, corresponds to the three highest levels of relative risk aversion of 29, 30, and 31. This means that an unconstrained investor with the level of risk aversion of 29 and

Table 1. Proportionate opportunity costs (θ-1) of constrained portfolios and composition of unconstrained portfolio

Relative risk aversion (1-γ)	Constrained portfolios: Opportunity cost of investing in stocks only	Constrained portfolios: Opportunity cost of investing in bonds only	Unconstrained portfolio: Optimal portfolio weights of stocks	Unconstrained portfolio: Optimal portfolio weights of bonds	Unconstrained portfolio: Optimal ratios of stocks to bonds
0.7	0.018	0.022	7.230	3.400	2.126
1	0.014	0.018	6.873	2.765	2.486
2	0.011	0.015	3.701	1.451	2.551
3	0.009	0.013	2.567	1.001	2.564
9	0.003	0.005	0.859	0.328	2.619
10	0.003	0.005	0.780	0.280	2.689
11	0.002	0.004	0.708	0.261	2.713
12	0.002	0.004	0.653	0.242	2.715
29	0.000	0.000	0.271	0.100	2.721
30	0.000	0.000	0.264	0.097	2.731
31	0.000	0.000	0.256	0.093	2.739

higher will be equally happy as if he was constrained. The highest proportionate opportunity cost, 1.8% (0.018), corresponds to the lowest level of relative risk aversion of 0.7; an unconstrained investor will be equally happy as if he was constrained to stocks only but had 1.8% more initial wealth.

Table 1 shows that as the level of relative risk aversion increases, both proportionate opportunity costs of investing in “only stocks” and “only bonds” portfolios decrease, given the CRRA utility function (1). The results indicate that optimal unconstrained portfolios offer high risk-tolerant investors broader, more daring investment opportunities. This kind of investors would require a premium to give up those investment opportunities.

What differs for the two types of constrained optimal portfolios is the magnitude of the proportionate opportunity costs. For the low levels of risk aversion, from 0.7 to 3, the values of the proportionate opportunity costs of investing in “only stocks” portfolios are lower than those for constrained optimal portfolios of “only bonds”. The optimal unconstrained portfolio shares for stocks and bonds in Table 1 for high risk-tolerant investors show that these investors follow very aggressive short-sell strategies in terms of Treasury bills by placing large proportions, larger than for bond indices, of their initial wealth in stock indices. Even though the constrained "only stocks" portfolios offer higher risk than constrained "only bonds" portfolio, they also offer higher expected returns than constrained "only bonds" portfolios (see Table 2). Therefore, "only bonds" portfolios are costlier than "only stocks" portfolios.

What is also interesting is the fact that for investors with risk aversion of nine through 12 the values of the proportionate opportunity costs of investing in "only stocks" or in "only bonds" are very close, and for investors with risk aversion of 29 and higher the proportionate opportunity costs are the same for the two constrained portfolio strategies. As investors become less and less risk-tolerant they place bigger and bigger proportions of initial wealth into Treasury bills. This change in portfolio weights makes stocks-and-Treasury-bills portfolios look more and more like bonds-and-Treasury-bills portfolios as risk aversion increases: portfolios with a large amount of initial wealth placed into Treasury bills, an equivalent of cash, and with a smaller amount of initial wealth placed into risky assets (either bonds or stocks).

The last column of Table 1 reports the ratios of optimal unconstrained portfolio shares of stocks to bonds. As relative risk aversion increases the optimal ratio of stocks to bonds increases very slightly – almost remaining constant. These results (slight increase in the optimal ratios) support Canner, Mankiw and Weil (1997) and can be explained as follows. As investors become less and less risk tolerant, the portion of initial wealth placed into Treasury bills increases (see Table 2). Given that returns on Treasury bills and bonds are highly correlated, according to Canner, Mankiw and Weil (1997), highly risk-averse investors will reduce the proportion of their initial wealth they place into bonds at a higher rate than that they place into stocks. Therefore, as risk aversion increases, the optimal ratio of stocks to bonds will increase also. But if we consider the optimal ratios from Table 1 as virtually constant, then these results are consistent with the mutual-fund separation theorem according to which the ratio of stocks to bonds is constant for investors with different levels of risk aversion. In both cases (the optimal ratios are slightly increasing or virtually constant) the empirical results do not agree with the popular advice according to which more risk-averse investors should hold a lower ratio of stocks to bonds. But that might be due to the fact that the ratios calculated here compared two stock indices holdings to two bond indices holdings not including Treasury bills. Another reason for my empirical results not matching the popular advice might be that our current understanding of investors' objectives might not be entirely accurate.

Table 2. Optimal portfolio shares for low, medium and high relative risk aversion

Relative risk aversion (1-γ)	Treasury bills	Government long-term bonds	Corporate bonds	S&P 500	NASDAQ	Gross expected monthly return	Certainty equivalent
(1-γ) = 0.7							
Unconstrained	-9.630	2.151	1.249	10.303	-3.073	1.062	1.042
Constrained - stocks	-9.569	0.000	0.000	3.678	6.891	1.049	1.027
Constrained - bonds	-2.850	3.341	0.509	0.000	0.000	1.036	1.021
(1-γ) = 11							
Unconstrained	0.021	0.181	0.080	0.929	-0.221	1.004	1.001
Constrained - stocks	0.346	0.000	0.000	0.245	0.409	1.002	1.000
Constrained - bonds	0.731	0.230	0.039	0.000	0.000	1.001	1.000
(1-γ) = 31							
Unconstrained	0.651	0.048	0.045	0.318	-0.062	1.002	0.999
Constrained - stocks	0.749	0.000	0.000	0.080	0.171	1.000	0.999
Constrained - bond	0.906	0.080	0.014	0.000	0.000	0.999	0.999

Table 2 reports optimal portfolio shares for unconstrained and constrained portfolio strategies for three different levels of relative risk aversion: low (0.7), medium (11) and high (31). For optimal constrained and unconstrained portfolios for risk aversion of 0.7, more than 100% of initial wealth, w_0, is held in the nominally risky assets, stock and bond indices, and Treasury bills are held in negative quantities. As risk aversion increases the proportion of initial wealth held in Treasury bills becomes positive and increases for optimal unconstrained and constrained portfolios.

Table 2 also shows gross monthly expected returns on unconstrained and constrained optimal portfolios, $E\ (X^{*\prime}\tilde{R})$. The net expected monthly portfolio return (gross expected monthly portfolio return minus 1, multiplied by 100%) for risk aversion of 0.7 is very dramatic for the unconstrained optimal portfolio (6.2%) and large for both constrained optimal portfolios (4.9% for "only stocks" and 3.6% for "only bonds" portfolios). Net expected returns are of small size for risk aversion of 11 and of 31. Such extreme magnitudes of expected portfolio returns for high risk-tolerant investors confirm the previously made conclusion about very aggressive short sale strategies. These magnitudes represent very leveraged portfolios (unconstrained as well as constrained). For investors with risk aversion of 11 and 31 there is some short selling going on also (in terms of the NASDAQ index and only for the unconstrained portfolios), but not as aggressive as for investors with risk aversion of 0.7. The less aggressive short selling for medium or high risk aversion leads to lower mean return portfolios.

We see that unconstrained and constrained expected portfolio returns for risk aversion of 11 and of 31 are closer to each other than that for risk aversion of 0.7. This shows that as risk aversion increases, the more nearly indifferent the investor is between the unconstrained and constrained portfolio strategies.

Another appropriate measure of the proportionate opportunity cost is the ratio of the certainty equivalents (Brennan and Torous 1999). The certainty equivalent shows the amount of certain wealth that would be viewed with indifference by an investor relative to having an uncertain amount of wealth. Accordingly, consider the ratio of the certainty equivalents of the unconstrained and constrained portfolios.

Denote the certainty equivalent by *CE*. The certainty equivalent (*CE*) is implicitly defined by

$$\frac{1}{\gamma}\ CE^{\gamma} = \frac{1}{\gamma}\ w_0^{\gamma}\ E\left(\tilde{R}^{\gamma}\right) \tag{20}$$

and so, with w_0=1,

$$CE = \left(E\left[\tilde{R}^{\gamma}\right]\right)^{\frac{1}{\gamma}}. \tag{21}$$

The certainty equivalent represents the amount of certain wealth that would be viewed with indifference relative to the optimal portfolio. Table 2 shows that as risk aversion increases the value of the certainty equivalent decreases (for the unconstrained portfolio strategy as well as for the constrained). As investors become more intolerant of risk they use less risky portfolio strategies and will be expecting lower returns from those portfolios, therefore, the certain amount of wealth they will be willing to accept with indifference will decrease.

Table 3. Percentage of certainty equivalent lost due to the "only stocks" and "only bonds" constraints

Type of assets	Relative risk aversion (1-γ)										
	0.7	1	2	3	9	10	11	12	29	30	31
Stocks only	1.6%	1.1%	0.9%	0.6%	0.3%	0.3%	0.2%	0.2%	0.0%	0.0%	0.0%
Bonds only	1.9%	1.6%	1.1%	0.8%	0.4%	0.4%	0.3%	0.3%	0.0%	0.0%	0.0%

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Table 3 presents the percentage of gross certainty equivalent for unconstrained portfolio strategies lost due to the constraint of investing in "only stocks" portfolios and to the constraint of investing in "only bonds" portfolios, computed as shown in (22). The percentage of gross certainty equivalent lost due to the constraints is timeless, as is the proportionate opportunity cost, θ-1.

$$Percentage\ loss = \frac{CE^*_{B\&S} - CE^*_{B\ or\ S}}{CE^*_{B\&S}} x100 = \frac{\theta - 1}{\theta} x100 \tag{22}$$

The highest percentage loss, 1.9%, occurs for the investors with risk aversion of 0.7 holding "only bonds" in their portfolios. As risk aversion increases, for both "only stocks" and "only bonds" portfolios, the percentage loss decreases. The lowest percentage loss, 0.0%, is observed for the investors with risk aversion of 29 and higher holding either "only stocks" or "only bonds" in their portfolios.

B. Proportionate regret

Large negative and positive asset holdings (Table 2) in portfolios for investors with a level of risk aversion of 0.7 suggest that the investors assume on a lot of risk. This raises the question: If the worst possible portfolio outcome occurs, then how much will the investors suffer from such an outcome? It is possible to measure the investors' proportionate regret from the worst-case scenario with such a risky portfolio.

Table 4. Ex-post proportionate regret (θ-1) under the worst portfolio outcome

Relative risk aversion (1-γ)	Bonds and stocks	Portfolios of only bonds	Only stocks
0.7	5.031	1.761	2.083
1	4.135	1.561	1.831
2	3.561	1.409	1.714
3	2.893	1.283	1.493
9	0.234	0.145	0.161
10	0.211	0.133	0.149
11	0.194	0.121	0.133
12	0.171	0.116	0.124
29	0.071	0.044	0.051
30	0.065	0.033	0.048
31	0.063	0.031	0.046

Table 4 reports the proportionate regret, (θ - 1), for unconstrained and constrained portfolio strategies, that will be incurred by investors if the worst possible outcome of asset returns occurs. This θ is defined by

$$U(\theta\,(X^{*\prime} R)^{worst}) = EU(X^* \tilde{R}), \tag{23}$$

where X^* is the optimally chosen portfolio, $(X^{*\prime}R)^{worst}$ is the one of the 120 states of nature giving the lowest portfolio return, $U[(X^{*\prime}R)^{worst}]$ is an investor's utility from getting the worst possible portfolio outcome, and $EU(X^{*\prime}\tilde{R})$ is an investor's ex ante expected utility.

The highest proportionate regret corresponds to the investors of low level of risk aversion of 0.7 with risky asset allocations. Those asset allocations are so risky that if the worst possible outcome occurs it would require investors to receive 503.1% of initial wealth in compensation in order to get the same level of ex post utility as their ex ante expected utility. For the high level of 31 for risk aversion, the proportionate regret for investors is 6.0% of their initial wealth. Such a low proportionate regret suggests that low risk-tolerant unconstrained investors choose very conservative unconstrained asset allocations.

For "only bonds" portfolios, the proportionate regret ranges from 176.1% (1.761) for risk aversion of 0.7, to 3.1% (0.031) for risk aversion of 31. For "only stocks" portfolios, the proportionate regret ranges from 208.3% (2.083) for risk aversion of 0.7, to 4.6% (0.046) for risk aversion of 31. This means that portfolios constrained to have stocks only or bonds only have a very restrictive character and do not let high risk-tolerant investors assume a lot of risk. For low risk-tolerant investors "only bonds" and "only stocks" portfolios are somewhat close to unconstrained (bonds and stocks) portfolios, and represent very conservative asset allocations with very little risk.

C. Opportunity cost with extreme values of returns exaggerated

In order to check the robustness of the estimates of the proportionate opportunity cost, results are re-calculated for the original data set with extremely high and extremely low simulated asset returns included. Simulated extremely high and extremely low asset returns are constructed the following way. For the original data set for each historical time period, compute the average excess return across all assets in the data set. The historical time period with the highest average excess return across all assets defines the historical period with the highest returns. The historical time period with the lowest average excess return across all assets defines the historical period with the lowest returns. Then, for the extreme historical periods only, calculate the deviation of each asset's return from that asset's intertemporal mean return. The deviations are doubled and then added back to assets' intertemporal means. This creates two fictional time periods with exaggerated high and exaggerated low returns. These fictional asset returns provide simulated extreme time periods to replace the time periods from which they were constructed. The rest of the original data set remains unchanged.

Using the "fictional" data set, repeat the whole procedure of calculating the proportionate opportunity cost as described in (6)-(19). The results from this computation are reported in Table 5.

Table 5. Proportionate opportunity costs (θ-1) of constrained portfolios and composition of unconstrained portfolio with extreme values of returns exaggerated

	Constrained portfolios		Unconstrained portfolio		
Relative risk aversion (1-γ)	Opportunity cost in stocks only	of investing in bonds only	Optimal portfolio weights of stocks	of bonds	Optimal ratios of stocks to bonds
0.7	0.039	0.046	5.103	2.320	2.199
1	0.029	0.037	4.189	1.675	2.501
2	0.023	0.031	2.514	0.976	2.576
3	0.017	0.026	1.871	0.713	2.623
9	0.007	0.011	0.633	0.237	2.671
10	0.007	0.011	0.549	0.204	2.693
11	0.006	0.009	0.541	0.200	2.731
12	0.006	0.009	0.532	0.193	2.763
29	0.002	0.004	0.333	0.120	2.779
30	0.002	0.004	0.325	0.116	2.804
31	0.002	0.003	0.311	0.111	2.806

The results in this second computation confirm the results obtained originally (Table 1): as the level of relative risk aversion increases, both proportionate opportunity costs of investing in “only stocks” and “only bonds” portfolios decrease. The low proportionate opportunity cost at high levels of risk aversion means that as the level of risk aversion increases, as investors become less risk tolerant, their perceptions of the optimal constrained portfolio strategy, “only bonds” and “only stocks”, even with extreme historical periods, are more similar to each other than perceptions of the strategy for investors with lower risk aversion.

The only difference between Table 1 and Table 5 is that the magnitudes of the proportionate opportunity cost are bigger when extreme returns are exaggerated. So the presence of exaggerated extreme returns in the probability distribution moves constrained portfolio optimization further from optimal and the decision of choosing a constrained portfolio over the unconstrained portfolio becomes more costly. This will increase the size of the cost that the investor will be incurring from choosing a constrained portfolio strategy, “only bonds” or “only stocks”, instead of the optimal strategy.

The results from this second computation (Table 5) also confirm another original conclusion (Table 1): as relative risk aversion increases, the optimal ratio of stocks to bonds increases very slightly, but remains virtually constant.

V. Conclusion

I have investigated the opportunity cost incurred by investors when they are constrained to invest in either stocks or bonds, instead of being unconstrained and investing in both stocks and bonds. The original historical asset returns are used, along with CRRA utility functions, to determine the proportionate opportunity cost. The opportunity cost has been calculated for different values of relative risk aversion (including extreme levels of relative risk aversion) for "only stocks" and "only bonds" portfolios for the original historical asset returns data set and for the historical asset returns data set with extreme returns exaggerated. I found for both types of constrained optimal portfolios that as the level of relative risk aversion increases, the proportionate opportunity cost decreases. The proportionate opportunity costs are bigger for the data set in which extreme returns are exaggerated. This can be explained by the fact that the presence of extreme returns in the probability distribution moves the constrained portfolios further away from the optimal portfolio, and the decision to adopt a constrained portfolio, rather than the unconstrained portfolios, becomes more costly.

The only difference between estimates of the proportionate opportunity cost for the two constrained portfolios is the magnitude of the estimates for both data sets. They are larger for the level of risk aversion of 0.7 for the "only bonds" portfolios. However, the difference in these opportunity costs is slight.

The present findings on optimal ratios of stocks to bonds for different levels of risk aversion confirm the mutual-fund separation theorem. The results indicate that as the level of risk aversion increases, the optimal ratio of stocks to bonds stays virtually constant.

An important conclusion from the analysis is that investors with low levels of risk tolerance will not benefit from full asset market participation: they do not incur any costs if they constrain themselves to only one type of assets in their portfolios. The absence of opportunity costs suggests that constrained portfolios are in fact investors' optimal portfolio choice at high levels of risk aversion. Empirical studies on household portfolio choice (Haliassos and Bertaut 1995, Bertaut 1998) indicate a very high rate of stock market non-participation. This may be explained by the hypothesis that most households are highly risk-averse. The results presented here show that the optimal portfolio strategy for very risk-averse investors is the "only bonds" portfolios.

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