Essays on Political Economy and Social Choice

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September 2003

[1]This thesis could not be written without the huge support of my parents, Raúl and Nélida, and my sister, Patricia. Therefore, it is primary dedicated to them. I am also indebted to Jorge Streb, Fernando Tohmé and Edgardo Zablotsky, for their help, guidance and advise during many years, and to Salvador Barberà, Jordi Massó and Alejandro Neme, for the stimulating and useful discussions held when I was a Marie Curie Fellow at the Universitat Autònoma de Barcelona. Finally, a very special recognition has to be made to Ana Paula, for her understanding and kindness, and for her love, which makes my life more valuable.

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Abstract

This thesis studies manipulation in policy-making processes. The first part focuses on decentralized collective choice environments, where individuals, best described as voters, could misrepresent their preferences to achieve better social states. The second part deals with pre-electoral manipulations of fiscal policy. In both parts a different aspect of the political process is considered. While the former searches for "minimum consensus", that is, restricted domains of individual preferences, that guarantees the existence of strategy-proof social choice mechanisms, the latter emphasizes the role of institutions, namely, separation of powers, to achieve more desirable policy outcomes.

JEL Classification: D70, D71, D78.

Keywords: Strategy-proofness; single-crossing; order-restriction; median voter; agenda-setter; political budget cycles; separation of powers; checks and balances; budgetary process.

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Chapter 1

Introduction

Manipulation of collective decision-making procedures and institutions is a major issue in economics. This thesis analyzes manipulation from two different perspectives. The first two chapters are developed into the general framework of social choice theory. They are concerned with manipulation in decentralized collective decision-making environments, where individuals, best described as voters, could misrepresent their preferences to achieve better social states. The last chapter, inserted into the literature on political economy, deals with pre-electoral manipulations of fiscal policy instruments.

Both parts put their accent on a different aspect of the political process. While the former searches for "minimum consensus", that is, restricted domains of individual preferences, that guarantees the existence of strategy-proof social choice mechanisms, the latter emphasizes the role of institutions, namely, separation of powers, to achieve more desirable policy outcomes.

In the rest of this chapter we summarize the problems to be studied and the main literature related with them.

1.1 Single-crossing preferences

In democratic societies, individual preferences must be the guide of all collective decisions. But knowledge about the values of concerned individuals is typically dispersed among them, and it is not obviously available (either directly or indirectly) to the decision-makers or to the institutions which determine social outcomes.

The following questions then arise naturally. Can we guarantee that agents will provide the decision-maker with accurate information regarding their preferences, or that they will behave within a given institution in a way that reveals their actual preferences? Could there be some institution for collective decision-making under which all individuals would always find it best to act straightforwardly and reveal their true preferences? Rules and institutional arrangements under which this would happen will be said to be non-manipulable by the individuals operating into them.

Strategy-proofness is a very strong form of robustness a social choice function may have against manipulation. A choice rule is called strategy-proof if telling the truth (i.e., if acting according to the true preferences) is a dominant strategy for every agent.

Such requirement is attractive, but hard to meet. A fundamental result, established by Gibbard (1973) and Satterthwaite (1975), states that a strategy-proof rule that is flexible enough to allow the possible election of at least three alternatives, must be dictatorial: there is an agent whose preferences dictate the final outcome irrespective of the other agents' reports.

The Gibbard-Satterthwaite Theorem holds under the so-called universal domain assumption, which implies that all possible preferences over the social alternatives are admissible for all agents. While this may be a natural assumption when the set of alternatives has no particular structure, it looks unreasonable strong when that set arises from some specific economic or political problems.

In many cases, the nature of the social decision problem induces a specific structure on the set of alternatives or (and) on the set of individuals, and this structure suggests, in turn, some restrictions on the set of admissible individual preferences. It is then natural to investigate whether the negative conclusion of the Gibbard-Satterthwaite Impossibility Theorem change when social choice functions are only required to operate on restricted domains of preferences.

Two types of preference restrictions, in particular, have been shown to give rise positive results.[1]. On the one hand, in economic contexts it is assumed that individuals care only about certain aspects of social alternatives. For instance, about public and own private consumption, but not about the distribution of the other individuals' private consumption. If, in addition, utility in private wealth is quasi-linear, the well-known class of Groves mechanisms offers a rich array of strategy-proof social choice functions.

By contrast, in pure social choice (voting) contexts individuals care about all aspects of the social state. Here, the assumption of single-peaked preferences not only arises naturally,[2] but also it has been proved to be an attrac-

[1]See Sprumont (1995) and Barberà (2001) for recent and comprehensive surveys of results characterizing strategy-proof collective choice rules when preferences satisfy conditions that are meaningful in economic and political environments.

[2]Preferences are single-peaked if alternatives can be linearly ordered, according to some criterion (from left to right, in political applications; from smaller to greater according to

tive domain for analyzing the manipulation of an aggregation mechanism. Effectively, in one of the most classical papers on this literature, Moulin (1980) showed that for this type of restricted domain there exists a wide class of strategy-proof social choice rules, the so-called generalized median voter schemes.

Single-peakedness, first discussed by Black (1948), makes sense in a broad variety of political and economic models. However, it faces at least two main limitations. First, there are settings in which assuming single-peaked preferences is not reasonable. For example, in the standard "one public good-one private good" model of public economics, if the public good production cost schedule is strictly concave, because there exist increasing returns to scale, then the induced preferences need not be single-peaked.

Secondly, social choice rules defined on single-peaked preferences may be strategy-proof in the unrestricted domain of alternatives, but not on arbitrary subsets of it. The reason is single-peakedness does not restrict too much the direction of preferences among alternatives that are not top. Thus, if agents were required to vote for their top on a given range, and their unconditionally best alternative were no longer available for some or all individuals, then there will be sufficiently room for manipulation (Barberà, et al. 1997).

Both criticisms limit the usefulness of single-peakedness for political and economic models, and they have encouraged economists to explore other families of individual preferences.

In the last two decades, two alternative preference domains have received increasing attention within the field of political economy. One of the main interests on these families is precisely that they were shown to be useful to ensure the existence of majority voting equilibria in situations where single-peakedness fails to hold.[3]

These conditions, while variously stated, are essentially the following. The first class requires a *single-crossing* property on the individuals' indifference curves being satisfied, which means that marginal rates of substitution must be monotone in some order of the individuals at all points in the choice

some quantitative index, in economic models; etc.), and individuals' preferences over this linear order of alternatives are such that, each individual has a unique ideal outcome; and individual's preferences are strictly decreasing as one moves away from his individual peak.

[3]For example, in Roberts' (1977) model, which analyzes the collective choice of redistributive tax and transfer rates schemes, single-peakedness may fail to account for the individual preferences, with the obvious consequence that the majority preference relation may be intransitive. In that case, Roberts used a condition he called hierarchical adherence, which implies order-restriction, to show that if it holds, there still exists a majority preference with the desired property of quasi-transitivity.

domain.[4] The second type consists of a more abstract condition requiring a global ordering of some representation of voters' preferences.[5] As it was said at the beginning, this family was first formally characterized by Rothstein (1990, 1991), and it is actually known as the *order-restricted* family of preferences.

Fortunately, Gans and Smart (1996) have unified these preference domains by showing that single-crossing and order-restriction are essentially equivalent conditions. Furthermore, they have also shown their relationship to the general, ordinal notion of single-crossing proposed by Milgrom and Shannon (1994), which is also connected with the literature of monotone comparative statics and to the more familiar Spence-Mirrlees single-crossing condition, frequently used in mechanism design, principal-agent theory and information economics.

The aim of the first part of this thesis will be to analyze the existence of non-trivial strategy-proof social choice functions on this preference domain,[6] and to characterize the sets of such functions when possible. That is, it will try to determine the complete class, if any, of strategy-proof social choice rules on preferences that satisfy single-crossing. Additionally, as a by-product of this research, it will also deal with the possibility of providing a game-theoretic (strategic) foundation for the Representative Voter Theorem of Rothstein (1991), the "order-restricted version" of the Median Voter Theorem.

We want to do this for several reasons. The first and more obvious one comes from the previous discussion, which makes clear the relevance of this family of preferences within economics and within other fields, like political economy and social choice. In addition, a second reason is related to the fact that single-crossing leads to analyze strategy-proofness on a preference domain where there exists a linear ordering of agents, rather than of alternatives. This not only contrasts with much of the work developed in this area, but also it looks particularly promising to study strategy-proofness in multi-dimensional choice spaces. Moreover, it also represents an attractive approach to deal with the manipulation of a choice rule in constrained domains of alternatives, a problem that has a great meaning in models of political economy.

Finally, as a by-product, these findings can also be used to study the existence of a non-cooperative strategic foundation of the Representative Voter Theorem (Rothstein, 1991). The importance of this Theorem resides in

[4]See, for example, Epple and Romer (1991), Westhoff (1977), and others.

[5]See, for example, Grandmont (1978), Roberts (1977) and others.

[6]We informally bunch up under the term "trivial" two types of rules: those that are dictatorial, and those which only choose between two alternatives.

that it offers a formal justification for a common technique, applied in many problems of collective decision-making with heterogeneous individuals. That technique consists in reducing the constituency to a single representative voter. (See chapter 4 for an application).

The problem with the Representative Voter Theorem is that, unlike the Median one, whose non-cooperative foundation was provided by Moulin (1980), the former is based on the assumption of *sincere voting*. Clearly, this assumption is difficult to maintain when the study focuses on policy choices taken in game-theoretic frameworks. Hence, a natural question related to its applicability in those settings arises. And this, by its own, gives another reason for our research.

1.2 Political budget cycles

One of the basic constitutional principles of liberal democracies is separation of powers. Since the writings of Locke and Montesquieu, separation of the legislative, executive and judicial powers is considered essential to avoid usurpation and tyranny by the holder of these powers.

Are separation of powers and "checks and balances" political arrangements significant to explain electoral distortions of fiscal policy? If so, in what sense? That is, which are the theoretical links, if any, among these institutional variables and the phenomena of political budget cycles?[7] More importantly, can these institutional aspects of the political system explain any systematic difference in the size and composition of political budget cycles in developed and developing countries? The second part of this thesis deals with these questions.

The creation of new and large cross-country databases of political institutions in recent years has renewed scholars' interest on comparative political economy. There are now three main databases of political institutions, which cover different time periods and different aspects of the political world: (1) The Database on Political Institutions, from the World Bank, (Beck, et at. 1999), provides data on 177 countries' political systems and elections, between 1975 and 1995. This database presents objective data at a disaggregated level; (2) Polity III, compiled by Gurr, Jaggers, and Moore (1998),

[7] Recent empirical works suggest that fiscal policy tends to be systematically manipulated before elections. These studies report evidence that shows changes in taxes, in aggregate spending and in spending composition. Moreover, they find these electoral cycles to be more pronounced in developing countries, ruled in most cases by worse democratic institutions. For further details, see Block (2002), Shi and Svensson (2002a, 2002b, 2003) and Schuknecht (1998), among others.

covers 156 nations for a time period stretching from 1800 to 1994. It includes a great number of subjective and highly aggregated indicators of the political and institutional environment; and, finally, (3) Henisz (1997), offers another valuable source of information on political constraints, particularly the data necessary to create a measure of checks and balances.

In particular, these developments have benefited the empirical literature on political business cycles. This literature, largely focused on a select group of developed countries,[8] has ignored in the past the role of political institutions for explaining the pre-electoral and post-electoral distortions on monetary and fiscal policy. The main reason for that was the lack of detailed data on political and institutional characteristics for a large sample of countries.

Things have become to change in the last years. There are now many papers that deal with these concerns. Shi and Svensson (2002a), for example, analyze a large panel of developed and developing countries (123 countries, for the period between 1975 and 1995), focusing on whether electoral cycles on government's budget balance interact with voters' access to information and private benefits (rents) politicians gain when in power. Using suitable proxies, they find that electoral budget cycles are common phenomena across countries; and that access to unbiased information and institutional and informal rules that constrain the incumbents from using public resources and policies for private gains reduce the magnitude of these cycles.[9]

On the other hand, using a data set encompassing sixty democracies from 1960 to 1998, Persson and Tabellini (2002) study how electoral cycles on fiscal policy differ across political systems. They find strong constitutional effects on the presence and nature of political budget cycles. According to their findings, when conditioning on electoral rules, pre-electoral spending cuts are more pronounced in majoritarian countries, while welfare-state spending rises before and after elections only in proportional countries. When conditioning on the form of government, they discover an intriguing difference between presidential and parliamentary countries. While pre-election tax cuts mainly take place in parliamentary systems, the post-election fiscal contractions take place only in presidential democracies. Finally, without conditioning at all on the political system, they find that taxes are cut before elections, painful fiscal adjustments are postponed until after the elections, while welfare-state spending displays no electoral cycle.

[8]See Alesina, et al. (1997), Drazen (2000a, 2000b) and Persson and Tabellini (2000), and the references found there, for surveys of earlier theoretical and empirical findings.

[9]They find that these institutional features differ markedly between the sample of developed and developing countries. Further, they show that fiscal deficits in the average developing country is significantly higher than in the average developed country, after controlling for per capita GDP and GDP growth rates.

Other paper in the same vein is Gonzalez (1999), which analyzes whether political budget cycles are related to the level of democracy in a country. In her work, the country's degree of democracy is given by the cost voters must bear when enforcing the political turnover after the election. She also characterizes the economy by an index of transparency, representing the likelihood with which voters learn the politician's competence. She shows not only that intermediate democracies generate pre-electoral fiscal distortions, but also that the magnitude of them has a "humped" shape relative to the degree of democracy prevailing in a country. She tests the model for Mexico's fiscal policy between 1957 and 1997. The estimation reveals the systematic use of public infrastructure by the government as an electoral tool. Further, the magnitude of the cycle is shown to depend on democracy and transparency, in concordance with her theoretical predictions.

Finally, Schucknecht (1998) studies a sample of 24 developing countries for the period between 1973 and 1992, reporting also evidence of electoral cycles in fiscal policy. And Block (2002) presents the first cross-country empirical analysis of electorally-motivated changes in the composition of public expenditures. His results show that election-year public expenditures shift towards more visible consumption goods, and away from public investment goods. The evidence also suggests that these effects are contingent on elections being competitive.

On the theoretical ground, the most recent literature on political budget cycles started with the signaling models of Rogoff (1990), Rogoff and Sibert (1988), and Persson and Tabellini (1990). Rogoff (1990), for example, showed that, by shifting government expenditures towards easily observed consumption spending and away from investment, the incumbent can signal his competence and increase his chances of reelection. We follow a similar approach in chapter 5, but we change the timing of the game, by assuming that incumbent's competence is realized after all agents (included the incumbent) have made their choices. This greatly simplify the analysis, and removes also the uncomfortable result that only competent types distort the economy, and that only them are reelected.

Finally, theoretical studies of separation of powers within economics has started to grow only recently. Persson, Roland and Tabellini (1997) constitutes one of the main papers in this literature. In general, their main conclusion is that political accountability is easier to achieve if the governing constitution allocates certain control rights to separate political offices. That is, accountability emerges only if it is clear who is responsible for an observed abuse of power. Instead, the benefits of separation of powers are lost if the offices can collude against the voters.

In chapter 5, we borrow many ideas from this paper, but to explain a different problem. More precisely, our model considers whether the details of the budgetary process and the constraints that they impose to the executive affect the size of the electoral distortion of the composition of government spending.[10] At least from our own knowledge of the literature, there is no previous research in this direction.

Similarly to other works in the area, we propose a moral-hazard model of electoral competition, in which policy outcomes (the composition of government spending) are influenced by the timing of the elections. Before elections, voters ignore the executive competence for providing public goods. They infer it from the policy implemented. The incumbent does not observe also his competence before selecting the amount of consumption and capital goods to be provided. But, in order to increase his chances of reelection, he faces an incentive to boost the supply of the more visible consumption goods, hoping that voters would attribute the boost to his competence. The strength of this incentive depends in part on the politico-institutional environment in which the politician acts.

Instead of assuming an "all-powerful" executive,[11] we introduce the novelty of two separate political offices, reflecting in a stylized way the existence of separation of powers and checks and balances in the budgetary process.[12] This implies that the executive is not anymore completely free to determine the per-period spending composition. Instead, with two policy-makers, the mix of public expenditures of each period must be agree upon by the two officials. And the product of this negotiation, the budgetary law, imposes limits on the discretion the executive enjoys at the implementation stage.

The intuition indicates that the greater are these limits, the lower will be the electoral distortion of the government spending composition. Further, it also suggests that the details of the budgetary process, the process by which the budgetary law is proposed, approved and implemented, and its deficiencies in many developing countries, could be a main factor to explain why there exists large differences between developed and developing countries in the size and composition of the electoral budget cycles.

Finally, the main motivation to carry out this work comes namely from the literature on budgetary institutions, which have been pointed out in the last

[10] Instead, Persson, et al. (1997) study whether the electorate can exploit the conflict of interests between the executive and the legislature to reduce the rents captured by politicians.

[11] This institutional setting can be thought as a democracy in which the executive controls 100 per cent of the seats of the legislature and has appointed all of the sitting justices of the Supreme Court.

[12] "Checks and balances" rules distribute proposal and veto rights to the politicians operating into the government.

decade their effects on fiscal performance.[13] These effects, we guess, should be particularly strong around elections, as long as weak institutions would imply higher discretion for the executive and, therefore, greater opportunities for election-oriented fiscal policies. In particular, strict implementation of the budgetary law and an independent control at this stage may be required to eliminate incentives for overspending and/or for shifting the spending composition in electoral periods. Although these ideas have a great appeal, they have been ignored for the theoretical models of political budget cycles. Thus, our main goal is to fill out this gap.

[13] See Alesina and Perotti (1995) for a survey of this literature. For empirical works, see for example Alesina et al. (1999) and Woo (2003), and the references quoted there.

Chapter 2

Single-crossing, strategic voting and the median choice rule

2.1 Introduction

In the last twenty five years, *single-crossing* has become a "popular" feature of preferences within the field of Political Economy.[1] From the seminal works of Roberts (1977) and Grandmont (1978) and, more recently, due to the theoretical developments of Rothstein (1990, 1991), Gans and Smart (1996) and Austen-Smith and Banks (1999), it is now well-known that this domain restriction is sufficient to guarantee the existence of equilibria in one-dimensional models of majority voting, especially in situations where single-peakedness may not hold.

Moreover, this restriction is not only technically convenient, but it also makes sense in many political settings. In few words, the single-crossing property used in the context of voting, which is similar to that used in principal-agent literature and monotone comparative statics, says that, given any two policies, one of them more to the right than the other, the more rightist is an individual (with respect to another individual) the more he will prefer the right-wing policy over the left-wing one.

Thus, unlike single-peakedness, single-crossing is a restriction that imposes limitations *across* individual preferences, on the character of voters' heterogeneity, rather than on the shape of individual preferences. The main idea behind it is that, in many circumstances, *ordering* people according to a single parameter (like income, productivity, intertemporal preferences, ideological position, etc.) may be more natural than ordering alternatives. Hence, under this condition, the conflict of interests among individuals is

[1] See, for example, the different applications found in Persson and Tabellini (2000).

assumed to be projected into a one-dimensional parameter space, and then the *types* of the agents are assigned a position over this left-right scale with the requirement that, for any pair of alternatives, the set of types preferring one of the alternatives all lie to one side of those who prefer the other.

It turns out that this condition not only guarantees the existence of majority voting equilibria, but it also provides a simple characterization of the core of the majority rule. In fact, the core is simply the set of ideal points of the median type *agent* in the ordering of individuals with respect to which the preference profile is single-crossing.[2] This result is sometimes referred to in the literature as the *Representative Voter Theorem* (Rothstein, 1991) (henceforth RVT) or, alternatively, as "the second version" of the Median Voter Theorem (Myerson, 1996 and Gans and Smart, 1996).

The main problem with this result is that, unlike the *original* Median Voter Theorem over single-peaked preferences, whose non-cooperative foundation was provided by Black (1948), first, and then by Moulin (1980), the RVT is based on the assumption that individuals honestly reveal their preferences. That is, it is derived assuming *sincere voting.* Clearly, this assumption is difficult to maintain in applications that focus on policy choices made in strategic frameworks. Hence, a natural question arises respect to its applicability in those models.

This chapter studies the strategic foundation of the Representative Voter Theorem. As a by-product, it also considers the existence of non-trivial strategy-proof social choice functions on the domain of single-crossing preference profiles and over the non-negative real line. There are several reasons that justify to carry out this analysis. But the first and more important one is that, even though single-crossing is now largely used in models of collective decision-making, nothing has been said in the literature about the possibility of manipulation over this domain. In particular, people uses the "single-crossing version" of the Median Voter Theorem without caring much about its strategic foundation. So, one of the main purposes here is to fill out this gap.

In addition, the study is also motivated by a more technical fact, though not less important. The analysis of strategic voting in the context of single-crossing preferences leads to consider strategy-proofness over a preference domain where there exists a linear ordering of the types of the agents and, therefore, a specific kind of *correlation* among individual preferences. This contrasts with much of the work developed in the field, which focuses on social

[2] In contrast, under single-peaked preferences, the core of the majority rule consists of the median *ideal points* in the ordering of alternatives with respect to which the profile is single-peaked.

choice rules defined over Cartesian preference domains. Moreover, this feature looks interesting for studying manipulation in multi-dimensional choice spaces and over constrained sets of alternatives, a problem that is extremely important in Political Economy (since voters usually have to choose from sets with only a few policies, rather than from the full set of alternatives).

The main result of the chapter shows that single-crossing preferences constitute a domain restriction in the real line that allows not only majority voting equilibria, but also non-manipulable choice rules. In particular, this is true for the median choice rule, which is found to be strategy-proof and group-strategic-proof not only over the full set of alternatives, but also over every possible policy *agenda*. This chapter also shows the close relation between single-crossing and order-restriction. And it uses this relation together with the strategy-proofness of the median choice rule to prove that the collective outcome predicted by the Representative Voter Theorem can be implemented in dominant strategies through a simple mechanism. This mechanism is a two-stage voting procedure in which, first, individuals select a representative among themselves, and then the representative voter chooses a policy to be implemented by the planner.

The chapter is organized as follows. Section 2.2 presents the model, the notation and the definitions. Section 2.3 exhibits the equivalence between single-crossing and *order-restriction* for preferences indexed by the types of the agents. Section 2.4 presents the non-strategic version of the Representative Voter Theorem (the “order-restricted version” of the Median Voter Theorem). The results related to strategy-proofness and the *indirect* implementation of the median choice rule over single-crossing preferences are presented in section 2.5, which also uses these and the results of section 2.3 to derive, as a by-product, the game-theoretic counterpart of the Representative Voter Theorem. The consequences of these results and further lines of research that stem from them are discussed in section 2.6.

2.2 The model, notation and definitions

The basic model of single-crossing preferences assumes that the set of agents I is finite and its cardinality $|I| = n > 2$ is odd. Individuals in I must choose a policy (for example, the level of a public good) from a feasible set of alternatives. They do this by voting.

The set of all possible collective outcomes $X = \{x_1, \ldots, x_l\}$, $|X| > 2$, is assumed to be a finite subset of the non-negative real line $\Re_+$. The set X is such that $x_j \leq x_k$ for $j \leq k$, where the linear order $\leq$ is the usual order on $\Re_+$. For a vector $x = (x_1, \ldots, x_n) \in \Re^n_+$, we let $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$

and $(\hat{x}_i,\, x_{-i}) = (x_1, \ldots, x_{i-1},\, \hat{x}_i,\, x_{i+1}, \ldots, x_n)$, where $\hat{x}_i \in \Re_+$. In addition, for any group of agents $D \subseteq I$, we denote $(x_D,\, x_{D^c}) = ((x_i)_{i \in D},\, (x_j)_{j \in D^c})$, where $D^c = I \backslash D$.

The set of all feasible alternatives may be either the entire X or just one of its non-empty subsets. The set $\tilde{X}$ represents a generic subset - with the induced order - of X. We use $A(X)$ to represent the set of all non-empty subsets of X, $A(X) = \{\tilde{X} : \tilde{X} \in 2^X \backslash \emptyset\}$. In words, X is the universal set of outcomes, whereas a particular situation, or *agenda*, involves a $\tilde{X} \in A(X)$.

Let $P(X)$ be the set of all complete, transitive and antisymmetric binary orderings of X. We say $P(X)$ is the *universal domain* of individual preferences.[3] Agent i's preferences over the alternatives in X are assumed to be completely characterized by a single parameter $\theta_i \in \Theta = \{\theta^1, \ldots, \theta^m\}$, where $\Theta \subset \Re$ is a finite and *ordered* subset of the real line, such that $\theta^1 < \theta^2 < \ldots < \theta^m$ and $m \leq |P(X)|$. As usual, we interpret θ_i as being agent i's *type*.

That is, we assume there exists a function $\Phi : \Theta \to P(X)$ that assigns a unique element $\succ_\theta \in P(X)$ to each $\theta \in \Theta$. We say that $\succ_i$ represents the preferences of an agent i of type θ_i if,

$$\forall x,\, y \in X,\; x \succ_i y \;\Leftrightarrow\; x\, \Phi(\theta_i)\, y.$$

The following example illustrates how these preferences can arise naturally in many political-economic settings:

Example 1 *(Persson and Tabellini, 2000) Consider the following simplified version of the redistributive distortionary taxation model of Roberts (1977). Suppose individual $i \in I$ has preferences $w(c_i, l_i) = c_i + v(l_i)$, $v'(\cdot) > 0$, $v''(\cdot) \leq 0$, where c_i denotes individual consumption and l_i leisure. The individual's budget constraint is $c_i \leq (1-t)h_i + f$, where $0 < t < 1$ is an income tax rate, f represents a lump-sum transfer and h_i is the individual labor supply. Individuals are heterogenous in a productivity parameter $\theta_i \in \Theta \subset \Re$, which is distributed in the population with mean $\bar{\theta}$. Given these different productivities, each individual i faces an "effective" time constraint $1 - \theta_i \geq l_i + h_i$. Finally, it is assumed that the government runs a balanced budget; i.e., $f \leq t\,(\sum_i h_i/n)$. Solving the model, we have that the induced policy preferences of agent i over alternative tax rates are*

$$u_i(t) = u(t\,; \theta_i) = h(t) + v[1 - h(t) - \bar{\theta}] - (1-t)(\theta_i - \bar{\theta}),$$

where $h(t) = 1 - \bar{\theta} - v_l^{-1}(1-t)$ is the average labor supply.

[3] Indifference between alternatives is not allowed. This is a natural assumption when the set of alternatives is finite.

The maximal set associated with the pair $\langle X, \succ_i \rangle$ is $M(X, \succ_i) = \{x \in X : \forall y \in X \backslash \{x\},\, x \succ_i y\}$. That is, $M(X, \succ_i)$ yields the alternative that is top-ranked in X for i with respect to her preferences $\succ_i$. Notice that since preferences are strict, maximal sets are indeed singletons.

A preference profile associated to a profile of types $\theta = (\theta_1, \ldots, \theta_n) \in \Theta^n$ is an n-tuple $(\succ_1, \ldots, \succ_n) = (\Phi(\theta_1), \ldots, \Phi(\theta_n))$ in $P(X)^n$. This means the profile of individual preferences depends on the *state* $\theta \in \Theta^n$: in the state θ, agent i has preferences $\Phi(\theta_i)$ over the set X. This formulation allows for any degree of correlation across the agents' preferences. We assume each agent observes θ, so there is complete information among the agents about their preferences over X. Extending our earlier conventions to preference profiles, we have $\succ_{-i} = (\succ_1, \ldots, \succ_{i-1}, \succ_{i+1}, \ldots, \succ_n)$. Similarly, the profile obtained by changing agent i's preferences for $\hat{\succ}_i$ is $(\hat{\succ}_i, \succ_{-i}) = (\succ_1, \ldots, \succ_{i-1}, \hat{\succ}_i, \succ_{i+1}, \ldots, \succ_n)$. Finally, for any group of agents $D \subseteq I$, $(\succ_D, \succ_{D^c}) = ((\succ_i)_{i \in D}, (\succ_j)_{j \in D^c})$.

Now, we restrict the set of admissible preference profiles by imposing a condition on preferences that involves the entire profile:

Definition 1 *A preference profile $(\succ_1, \ldots, \succ_n)$ derived from $\Phi : \Theta \to P(X)$ is single-crossing on X if, for all x, $y \in X$ and all i, $j \in I$ such that either $y > x$ and $\theta_j > \theta_i$ or $y < x$ and $\theta_j < \theta_i$,*

$$y \, \Phi(\theta_i) \, x \quad \Rightarrow \quad y \, \Phi(\theta_j) \, x. \qquad SC$$

We denote $SC(X)$ the set of all single-crossing preference profiles on X.[4] The recent interest on this restricted domain of preferences is due to the fact that, like single-peakedness,[5] single-crossing has been shown to be sufficient to guarantee the existence of majority voting equilibria. However, apart from this fact, it should be clear that both domain conditions are independent, in the sense that neither property is logically implied by the other. In Example 1, for instance, it is easy to see that the profile of induced policy preferences $(u_1, \ldots, u_n)$ satisfies single-crossing. However, for $h(t)$ sufficiently convex, it violates single-peakedness. (See also Examples 2 and 3 below.)

[4]Other expressions used in the literature to denominate this preference restriction are *hierarchical adherence*, *order-restriction* and *unidimensional alignment*. For more on them, see Roberts (1977), Rothstein (1990, 1991), Gans and Smart (1996), Austen-Smith and Banks (1999) and List (2001), and the references quoted there.

[5]Formally, a preference profile $(\succ_1, \ldots, \succ_n)$ is single-peaked on X with respect to the linear order $\leq$ if for all $i \in I$, there exists $\tau_i \in X$, called the *peak* of i associated to the preference relation $\succ_i$, such that (1) $\tau_i \succ_i x$, for all $x \in X \backslash \{\tau_i\}$; (2) $y < x \leq \tau_i$ implies $x \succ_i y$, and (3) $\tau_i \leq x < y$ implies $x \succ_i y$.

Furthermore, from the perspective of the analysis of strategy-proofness, there is a huge difference among these two preference domains. While single-peaked profiles of individual preferences define a subset of $P(X)^n$ that constitutes a Cartesian product, single-crossing profiles do not. That is, $SC(X)$ cannot be written as a Cartesian-product preference domain. The reason is that individual preference orderings (or types) in $(\succ_1, \ldots, \succ_n) \in SC(X)$ are correlated, in the sense specified in Definition 1, instead of being completely independent of each other.

As we will see, this implies that, even if a social choice function (yet to be defined) is strategy-proof on $SC(X)$, a mechanism implementing it has to be more complex than a straightforward one. We will return to this point in the last section of the paper. For the moment, let us illustrate how these preferences look like through the following two examples:

Example 2 *Suppose there are three types (each of them possibly associated to a group of individuals), indexed* $\theta_1 < \theta_2 < \theta_3$*, who must choose an alternative from the finite subset* $\{x, y, z\} \subset \Re_+$*,* $x < y < z$*. Assume that the types have the preferences depicted in Table 2.1 below.*

Table 2.1: Example 2

$\Phi(\theta_1)$	$\Phi(\theta_2)$	$\Phi(\theta_3)$
x	x	z
y	z	y
z	y	x

It is easy to see that this profile is single-crossing on $\{x, y, z\}$*. However, for any ordering of the alternatives, the profile violates single-peakedness.*

Example 3 *Suppose three individuals,* 1*,* 2 *and* 3*, that have to choose an alternative from the subset* $\{a, b, c, d\} \subset \Re_+$*. Assume their preferences* $\succ = (\succ_1, \succ_2, \succ_3)$ *are as in Table 2.2. Then, the profile* $\succ$ *is single-peaked with respect to the ordering of the alternatives* $c < a < b < d$*. However, if each individual* i *is associated to a type* θ_i*, it violates single-crossing.*

In the political arena, single-crossing makes sense if, for example, individual types are interpreted as being different ideological characters, arranged in the left-right scale, and the alternatives as public policies to be chosen by the society. Put in this way, it says that, given any two policies, one of them

Table 2.2: Example 3

$\succ_1$	$\succ_2$	$\succ_3$
a	d	b
b	b	a
d	a	c
c	c	d

more to the right than the other, the more rightist a type the more will he prefer the right-wing policy over the left-wing one.[6]

Given a preference $\succ_i$ in the profile $\succ \in SC(X)$, we define agent i's *induced* preferences over the agenda $\tilde{X} \in A(X)$, $\tilde{\succ}_i$, as follows:

$$\forall x,\, y \in \tilde{X},\ x \,\tilde{\succ}_i\, y \ \Leftrightarrow\ x \succ_i y.$$

Notice that the property of being single-crossing is preserved in the induced preferences. That is, if $\succ \in SC(X)$ then $\tilde{\succ} \in SC(\tilde{X})$, for all $\tilde{X} \in A(X)$.

These preferences can be aggregated. The input for this aggregation process is the set of *declarations* of the individuals. These declarations are intended to provide information about their true types, although their sincerity cannot be ensured.

The aggregation process is represented by a social choice function. For any $\tilde{X} \in A(X)$, a *social choice function* f on $SC(\tilde{X})$ is a single-value mapping $f : SC(\tilde{X}) \to \tilde{X}$ that associates to each preference profile $\tilde{\succ} = (\tilde{\succ}_1, \ldots, \tilde{\succ}_n) \in SC(\tilde{X})$ a unique outcome $f(\tilde{\succ}) \in \tilde{X}$.

We are primarily interested in aggregation procedures conducted by pairwise majority voting. This rule leads in the domain of single-crossing preferences and under the assumption of *sincere* voting to a collective outcome that coincides with the median type agent's most-preferred alternative (see Theorem 1 below). We will examine in the next sections if agents, endowed with this kind of preferences, have incentives to misrepresent their types in the aggregation process. But first, we need to define some additional concepts.

[6]Notice the difference with single-peakedness: "Intuitively, a single-peaked profile is one in which the set of alternatives can be ordered along a left-right scale in such a way that each individual has a unique most-preferred alternative (or *ideal point*) and the individual's ranking of other alternatives falls as one moves away from her ideal point. Such profiles capture the common intuition that, for example, an individual has a most preferred ideological position on some liberal-conservative spectrum and the more distant is a candidate's ideological position from this most-preferred point the more the individual dislikes the candidate." (Austen-Smith and Banks (1999), pp. 93.)

For any odd positive integer k, let $m^k : \Re_+^k \rightarrow \Re_+$ be the *k-median function*, defined in the following way: for all $x \in \Re_+^k$, $m^k(x)$ is the *k-median* of $x = (x_1, \dots, x_k)$ if and only if $|\{x_i \in \Re_+ : x_i \leq m^k(x)\}| \geq \frac{(k+1)}{2}$ and $|\{x_j \in \Re_+ : m^k(x) \leq x_j\}| \geq \frac{(k+1)}{2}$. Because k is odd, this function is always well-defined.

Now, we define the *median choice rule* in the following way. For any individual ordering $\tilde{\succ}_i$ in $\tilde{\succ} \in SC(\tilde{X})$, let $\tau(\tilde{\succ}_i) = M(\tilde{X}, \tilde{\succ}_i)$:

Definition 2 *A social choice function f^m on $SC(\tilde{X})$ is called the median choice rule if for all $\tilde{\succ} \in SC(\tilde{X})$,*

$$f^m(\tilde{\succ}) = m^n(\tau(\tilde{\succ}_1), \dots, \tau(\tilde{\succ}_n)).$$

A crucial property we seek in a social choice function is *strategy-proofness*, and the related concept of *group-strategy-proofness.* That is, we want to consider voting rules where agents, acting individually or in groups, never have the incentives to misrepresent their preferences. To capture this idea, we define the following two concepts:

Definition 3 *A social choice function f on $SC(\tilde{X})$ is strategy-proof if for all $\tilde{\succ} \in SC(\tilde{X})$, and for any agent $i \in I$, with type θ_i, any misrepresentation $\hat{\succ}_i = \tilde{\Phi}(\hat{\theta}_i)$, $\hat{\theta}_i \neq \theta_i$, is such that either $f(\tilde{\succ}) \, \tilde{\succ}_i \, f(\hat{\succ}_i, \tilde{\succ}_{-i})$ or $f(\tilde{\succ}) = f(\hat{\succ}_i, \tilde{\succ}_{-i})$, where $(\hat{\succ}_i, \tilde{\succ}_{-i}) \in SC(\tilde{X})$.*[7]

If a social choice function f is not strategy-proof, then there exist $i \in I$ and $\hat{\succ}_i$ such that for some $\tilde{\succ}_{-i}$, $(\hat{\succ}_i, \tilde{\succ}_{-i}) \in SC(\tilde{X})$, and i's true preferences, $\tilde{\succ}_i$, $f(\hat{\succ}_i, \tilde{\succ}_{-i}) \, \tilde{\succ}_i \, f(\tilde{\succ}_i, \tilde{\succ}_{-i})$. Then, we say f is *manipulable* at $(\tilde{\succ}_i, \tilde{\succ}_{-i})$, by i, via $\hat{\succ}_i$. In the same way:

Definition 4 *A social choice function f on $SC(\tilde{X})$ is group-strategy-proof if for all $\tilde{\succ} \in SC(\tilde{X})$, and for every coalition $D \subseteq I$, with types $\theta_D = (\theta_i)_{i \in D}$, there does not exist a joint misrepresentation $\hat{\succ}_D = (\tilde{\Phi}(\hat{\theta}_i))_{i \in D}$, $\hat{\theta}_D \neq \theta_D$, such that, for all $i \in D$, $f(\hat{\succ}_D, \tilde{\succ}_{D^c}) \, \tilde{\succ}_i \, f(\tilde{\succ})$, where $(\hat{\succ}_D, \tilde{\succ}_{D^c}) \in SC(\tilde{X})$.*

In the following sections, we will study how well the median choice rule performs, according to these manipulation criteria, on the domain of single-crossing preference profiles. But, since the main motivation to do this is to study the strategic foundation of the Representative Voter Theorem (the "single-crossing version" of the Median Voter Theorem), let us discuss first the connection between single-crossing and order-restriction, which is the original domain where this Theorem was formulated.

[7] With $\tilde{\Phi}(\cdot)$ we represent the restriction of $\Phi(\cdot)$ over $\tilde{X}$.

2.3 Single-crossing and order-restriction

Order-restriction, introduced formally for the first time by Rothstein (1990, 1991), is a preference restriction that has been shown to be closely related to single-crossing (Gans and Smart, 1996). Next we provide its definition and an *equivalence* theorem (up to renaming of types) that parallels that result, but that is more consistent with Rothstein's original characterization.[8]

For any two sets of integers A and B, let $A >_S B$, read "A is higher than B", if for every $a \in A$ and $b \in B$, $a > b$.

Definition 5 *A preference profile* $(\Phi(\theta_1), \ldots, \Phi(\theta_n)) \in P(X)^n$ *is order-restricted on* X *if and only if there exists a permutation* $\gamma : \Theta \rightarrow \Theta$ *such that for all distinct pair of alternatives* $x, y \in X$*, either*

$$\{\gamma(\theta) \in \Theta : x\,\Phi(\gamma(\theta))\,y\} >_S \{\gamma(\theta) \in \Theta : y\,\Phi(\gamma(\theta))\,x\} \qquad OR-1$$

or

$$\{\gamma(\theta) \in \Theta : y\,\Phi(\gamma(\theta))\,x\} >_S \{\gamma(\theta) \in \Theta : x\,\Phi(\gamma(\theta))\,y\} \qquad OR-2$$

We call $OR(X)$ the set of all order-restricted preference profiles on X. In words, a profile is order-restricted on X if we can order the types of the individuals in such a way that for any pair of alternatives the set of types preferring one of the alternatives all lie to one side of those who prefer the other. It is important to emphasize that the ordering of types is not conditional on the pair of alternatives under consideration, while the "cut-off" types may depend on the pair. Example 4 below illustrates the concept.

Example 4 *(Austen-Smith and Banks, 1999): Consider the preferences over* $X = \{x,\ y,\ z\}$*, with the order* $x < y < z$*, for the types* $\theta_1 < \theta_2 < \theta_3$*, displayed in Table 2.3. This profile is order-restricted over* X*, since there exists a permutation* γ*, defined by* $\gamma(\theta_1) = \theta_2$*,* $\gamma(\theta_2) = \theta_1$ *and* $\gamma(\theta_3) = \theta_3$*, such that under this renaming of types we have that:*

- $\{\theta :\ x\,\Phi(\theta)\,y\} = \{\theta_1, \theta_2\} <_S \{\theta_3\} = \{\theta :\ y\,\Phi(\theta)\,x\}$;
- $\{\theta :\ x\,\Phi(\theta)\,z\} = \{\theta_1, \theta_2\} <_S \{\theta_3\} = \{\theta :\ z\,\Phi(\theta)\,x\}$;
- $\{\theta :\ y\,\Phi(\theta)\,z\} = \{\theta_1\} <_S \{\theta_2, \theta_3\} = \{\theta :\ z\,\Phi(\theta)\,y\}$.

The following results exhibit the close relationship between OR and SC.

[8] In this section, we will make definitions and proofs over X, but everything is equally valid for any $\tilde{X} \in A(X)$.

Table 2.3: Example 4

$\Phi(\theta_1)$	$\Phi(\theta_2)$	$\Phi(\theta_3)$
x	x	z
z	y	y
y	z	x

Lemma 1 *If a preference profile $\succ$ derived from $\Phi : \Theta \to P(X)$ is single-crossing on X then, it satisfies order-restriction on X.*

Proof In order to show this, consider a profile $(\succ_1, \ldots, \succ_n) \in SC(X)$. Choose any x, $y \in X$ and, without loss of generality, assume $y > x$. Since Θ is finite, there exists $\theta^* \in \Theta$ such that $\theta^* = \min_{\theta} \{\theta \in \Theta : y\,\Phi(\theta)\,x\}$. If such type does not exist, then $x\,\Phi(\theta)\,y$ for all $\theta \in \Theta$ and order-restriction follows immediately. Otherwise, by single-crossing, $y\,\Phi(\theta)\,x$ for all $\theta > \theta^*$. Finally, by the completeness of the binary relation, $x\,\Phi(\theta)\,y$ for all $\theta < \theta^*$. Hence, $(\succ_1, \ldots, \succ_n) \in OR(X)$. $\square$

However, the converse is not true. Just consider the original ordering in Example 4. As we showed, it is in $OR(X)$, but it is not in $SC(X)$ as, for example, $z\,\Phi(\theta_1)\,y$ while $y\,\Phi(\theta_2)\,z$, being $z > y$ and $\theta_2 > \theta_1$. Nevertheless we have the following result:

Lemma 2 *For any profile $\succ$, derived from $\Phi : \Theta \to P(X)$, such that $\succ \in OR(X)$, there exists a permutation $\bar{\gamma} : \Theta \to \Theta$, such that the profile $\succ^{\bar{\gamma}}$, derived from $\Phi : \bar{\gamma}(\Theta) \to P(X)$, verifies $\succ^{\bar{\gamma}} \in SC(X)$.*

Proof Consider a preference profile $\succ \in OR(X)$. Since $\succ \in OR(X)$, there exists a permutation γ such that for $\gamma(\Theta)$ and any pair of alternatives x, $y \in X$, say $x < y$, we have either $OR-1$ or $OR-2$. In the latter case, consider $\theta^* \in \gamma(\Theta)$, such that $\theta^* = \min_{\theta} \{\theta \in \gamma(\Theta) : y\,\Phi(\theta)\,x\}$. Therefore, since $y\,\Phi(\theta^*)\,x$, we have that $y\,\Phi(\theta)\,x$, for any $\theta \in \gamma(\Theta)$ such that $\theta > \theta^*$. Thus, for $\bar{\gamma} = \gamma$, the profile $\succ^{\bar{\gamma}}$ is in $SC(X)$. Instead, if γ is such that for $x < y$ it verifies $OR-1$, consider a permutation $\gamma' : \gamma(\Theta) \to \gamma(\Theta)$, such that (if $|\gamma(\Theta)| = |\Theta| = m$), $\gamma'(\theta_i) = \theta_{m-i+1}$, for every $\theta_i \in \gamma(\Theta)$. This permutation just induces a reversion of the ordering in $\gamma(\Theta)$. Then, composing γ' and γ we have a permutation $\bar{\gamma}$ such that on $\bar{\gamma}(\Theta)$ we have $OR-2$ and again, $\succ^{\bar{\gamma}} \in SC(X)$. $\square$

Notice that this result amounts to an equivalence (under renaming of types in Θ) of SC and OR.

2.4 The Representative Voter Theorem

Single-crossing (order-restriction) has some properties that have been shown to be very useful in the analysis of collective decision-making processes. The first one, already mentioned in other parts of the chapter, is that it guarantees the existence of majority voting equilibria.

Additionally, it can also be shown that, when preferences are order-restricted, the *median type* agent in the order on $(\theta_1, \ldots, \theta_n)$ (which is unique in our framework since I is odd) is decisive in all pairwise majority contests between alternatives in $\tilde{X}$, for all $\tilde{X} \in A(X)$.[9] This result is sometimes referred to as the Representative Voter Theorem (RVT) or, alternatively, as the "second version" of the Median Voter Theorem.

In this section we will present formally the RVT, leaving for the next section the task of proving its game-theoretic counterpart. But first, two comments are in order. The first is to note that we will present only a simplified version of the original RVT. It is simpler because neither individual indifference nor the case with an even number of voters is considered.[10]

The second observation is that the original formulation and the proof of the RVT were given in the context of order-restricted preferences (see Rothstein, 1991). However, since we have shown the equivalence, under renaming of types, of order-restriction and single-crossing, we will exploit in the next section the fact that the median choice rule is strategy-proof over single-crossing preferences to prove the validity of the RVT in strategic environments. So, to maintain the internal consistency of the chapter, our proof here of the RVT uses the single-crossing condition, instead of order-restriction.

The non-strategic version of the Representative Voter Theorem is as follows:

Theorem 1 *Let $f^m : OR(X) \to X$ be the median choice rule on the domain of order-restricted preferences. Then, for each preference profile $\succ \in OR(X)$, and for every nonempty subset $\tilde{X} \in A(X)$, $f^m(\tilde{\succ}) = M(\tilde{X}, \tilde{\Phi}(\theta_r))$, where $\theta_r = m^n(\theta_1, \theta_2, \ldots, \theta_n)$.*

Proof Consider a preference profile $\succ \in OR(X)$. By Lemma 2, there exists a profile $\succ^{\bar{\gamma}} \in SC(X)$ that obtains by renaming the types $\{\theta_i\}_{i \in I}$. Take the agenda $\tilde{X} \in A(X)$ and the restriction of $\succ^{\bar{\gamma}}$ to $\tilde{X}$, $\tilde{\succ}^{\bar{\gamma}}$. Define the set of individuals' maximal alternatives in $\tilde{X}$ according to $\tilde{\succ}^{\bar{\gamma}}$ as

[9] See, for example, Rothstein (1991), Myerson (1996), Gans and Smart (1996), Austen-Smith and Banks (1999) and Persson and Tabellini (2000).

[10] For a more complete treatment, see the references listed in footnote 9.

follows: $T(\tilde{X}, \tilde{\succ}^{\bar{\gamma}}) = \{\tau(\tilde{\succ}_1^{\bar{\gamma}}), \ldots, \tau(\tilde{\succ}_i^{\bar{\gamma}}), \ldots, \tau(\tilde{\succ}_n^{\bar{\gamma}})\}$. We claim that, for all $i, j \in I$, if $\theta_i^{\bar{\gamma}} < \theta_j^{\bar{\gamma}}$, then $\tau(\tilde{\succ}_i^{\bar{\gamma}}) \leq \tau(\tilde{\succ}_j^{\bar{\gamma}})$. Suppose not. That is, assume by contradiction $\tau(\tilde{\succ}_i^{\bar{\gamma}}) > \tau(\tilde{\succ}_j^{\bar{\gamma}})$. Since $\tau(\tilde{\succ}_i^{\bar{\gamma}}) \tilde{\succ}_i^{\bar{\gamma}} \tau(\tilde{\succ}_j^{\bar{\gamma}})$ and $\theta_i^{\bar{\gamma}} < \theta_j^{\bar{\gamma}}$, by single-crossing, we have that $\tau(\tilde{\succ}_i^{\bar{\gamma}}) \tilde{\succ}_j^{\bar{\gamma}} \tau(\tilde{\succ}_j^{\bar{\gamma}})$. Absurd. Thus, the set $T(\tilde{X}, \tilde{\succ}^{\bar{\gamma}})$ has to be ordered from the lowest to the highest top; and, therefore, it follows that $f^m(\tilde{\succ}^{\bar{\gamma}}) = m^n(\tau(\tilde{\succ}_1^{\bar{\gamma}}), \ldots, \tau(\tilde{\succ}_n^{\bar{\gamma}})) = \tau(\tilde{\succ}_r^{\bar{\gamma}}) = M(\tilde{X}, \tilde{\Phi}(\theta_r^{\bar{\gamma}}))$, where $\theta_r^{\bar{\gamma}} = m^n(\theta_1^{\bar{\gamma}}, \theta_2^{\bar{\gamma}}, \ldots, \theta_n^{\bar{\gamma}})$. Finally, notice that $\theta_r^{\bar{\gamma}} = \theta_r$, where $\theta_r = m^n(\theta_1, \ldots, \theta_n)$, since, according to the proof of Lemma 2, $\bar{\gamma}$ is either the identity (meaning that, for each i, $\theta_i^{\bar{\gamma}} = \theta_i$) or it is a reversion of the original ordering (implying that, for each i, $\theta_i^{\bar{\gamma}} = \theta_{m-i+1}$). In either case, $m^n(\theta_1^{\bar{\gamma}}, \ldots, \theta_n^{\bar{\gamma}}) = m^n(\theta_1, \ldots, \theta_n)$. □

In words, Theorem 1 says that, given any subset of policies $\tilde{X} \in A(X)$, the alternative chosen by a society with order-restricted preferences is the most preferred option of the median type agent.[11] This result holds also under single-peakedness if individual preferences are symmetric, but not in other cases. Figure 2.1 below illustrates this point:

Figure 2.1: Median vs. representative voter

In the picture, preferences over the full set of alternatives, $X = [0, 1]$, are single-peaked. Therefore, the Median Voter Theorem applies, and agent 2's *unrestricted* top, τ_2, wins in pairwise majority voting. Moreover, the *induced* profile of preferences over the subset $\tilde{X} = \{a, b, c, d\} \subset X$ satisfies also single-peakedness, (along the linear ordering $c < a < b < d$).[12] However, it is not single-crossing. Then, it turns out that agent 2's most preferred alternative in $\tilde{X}$, d, is defeated by the alternative b, which is agent 3's *restricted* top and the Condorcet winner in $\tilde{X}$.

Thus, what this example shows is that under single-peakedness the median agent may depend on the particular agenda considered. This does not happen under single-crossing. Theorem 1 guarantees that the median type θ_r

[11] Rothstein (1991) has also shown that, when preferences are strict and the number of voters is odd, as in our case, the preference ordering induced by the majority rule coincides with the preference relation of the median type agent. This implies that the majority preference relation inherits all the properties of the median type agent's preference ordering. In particular, transitivity. Gans and Smart (1996) have proven a similar result for non-strict preference orderings, but under strict single-crossing.

[12] In fact, this is the profile introduced in Example 3 above.

(and hence the individual who is of this type) is decisive over any non-empty subset $\tilde{X} \in A(X)$.

However, is the collective outcome predicted by the RVT robust to individual or group manipulation? That is, can we expect this outcome to hold when voters act strategically? The Representative Voter Theorem is a result derived under the assumption that individuals honestly reveal their preferences or, alternatively, under the assumption that the decision-maker knows them. Both assumptions are obviously very strong.

Fortunately, it turns out that, even if we relax these assumptions, admitting both private information of individual values and strategic behavior on the part of voters, the RVT still holds. As we will see in the next section, the reason is that the median choice rule f^m is strategy-proof on the domain of single-crossing preference profiles. This implies that, in any majority contest, each agent has a dominant strategy, which is to honestly reveal his preferences. Therefore, the RVT applies, meaning that the outcome predicted by Theorem 1 must be expected no matter what strategic considerations are allowed. In the following section, we derive this result formally and we provide an *indirect* mechanism that implements the prediction of the RVT in dominant strategies.

2.5 Manipulation in single-crossing domains

The manipulation of the median rule has been studied for a long time in the literature of social choice. The earliest reference goes back to the seminal paper of Black (1948). Since then, a lot of progress has been made towards the understanding of its properties. For instance, it is well-known today that there exists a preference domain where this voting procedure performs quite well, in terms of its capacity to extract truthful information about the preferences of the agents. This domain is of course single-peakedness.

In this section, we analyze whether the median choice rule can be manipulated on a different preference domain, namely over single-crossing preferences. Even though this family of preferences is now largely used in models of collective decision-making process, nothing has been said in the existence literature about the possibility of manipulation over this domain. In particular, people uses the "single-crossing version" of the Median Voter Theorem without caring much about its strategic foundation. The main purpose here is therefore to fill out this gap.

Our main result is the following:

Proposition 1 *The median choice rule* f^m *is strategy-proof over* $SC(\tilde{X})$*, for any* $\tilde{X} \in A(X)$*.*

Proof Consider a profile $\tilde{\succ} = (\tilde{\succ}_i, \tilde{\succ}_{-i}) \in SC(\tilde{X})$, where agent i, of type θ_i, has preferences $\tilde{\succ}_i$. Suppose that there exists another type $\hat{\theta}_i$ such that $\hat{\succ}_i = \tilde{\Phi}(\hat{\theta}_i)$, $(\hat{\succ}_i, \tilde{\succ}_{-i}) \in SC(\tilde{X})$, and $f^m(\hat{\succ}_i, \tilde{\succ}_{-i}) \tilde{\succ}_i f^m(\tilde{\succ})$. Furthermore, without loss of generality, assume that $\tau(\tilde{\succ}_i) < f^m(\tilde{\succ})$. We have two cases to consider:

1. $\tau(\hat{\succ}_i) \leq f^m(\tilde{\succ})$. Then, $f^m(\hat{\succ}_i, \tilde{\succ}_{-i}) = f^m(\tilde{\succ}_i, \tilde{\succ}_{-i})$. Contradiction;

2. $\tau(\hat{\succ}_i) > f^m(\tilde{\succ})$. Then $f^m(\hat{\succ}_i, \tilde{\succ}_{-i}) > f^m(\tilde{\succ}_i, \tilde{\succ}_{-i})$. Let us call $\tilde{\tau} = f^m(\tilde{\succ}_i, \tilde{\succ}_{-i})$ and $\hat{\tau} = f^m(\hat{\succ}_i, \tilde{\succ}_{-i})$. Since we assume that $\tilde{\succ}$ verifies the single-crossing property, we have that $\hat{\tau}\, \tilde{\Phi}(\theta)\, \tilde{\tau}$ for all $\theta \geq \theta_i$. On the other hand, since $\tilde{\tau}$ is the maximal for at least one $\tilde{\succ}_j$ in $\tilde{\succ}$, it must be that the type corresponding to $\tilde{\succ}_j$, say θ_j, is such that $\theta_j < \theta_i$. But then, since $\tau(\tilde{\succ}_i) < \tilde{\tau}$, by single-crossing we have that $\tilde{\tau}\, \tilde{\Phi}(\theta)\, \tau(\tilde{\succ}_i)$ for every $\theta > \theta_j$. In particular for θ_i. Contradiction. □

Thus, Proposition 1 makes the important contribution of proving that, apart from single-peakedness, there exists another very *natural* preference domain over the real line where strategy-proof choice rules can be found. That is, it shows that single-crossing preferences constitute a domain restriction that allows not only majority voting equilibria, but also the existence of non-trivial strategy-proof social choice functions. In particular, this is true for the median choice rule.

In the next chapter we will show that the whole family of strategy-proof social choice functions over single-crossing preferences is given by a subclass of the *extended median rules*, obtained by distributing the *phantom voters* at the extremes of the non-negative real line. This subclass, where each phantom voter is either a *leftist* or a *rightist*, is sometimes referred to as *positional dictator* choice rules (see Moulin (1988), pp. 302). These rules select the kth ranked peak among the tops of the reported preference orderings, for some $k = 1, \ldots, n$. For example, if $k = 1$, we have the *leftist rule*, which chooses the smallest reported peak of a real voter. Of course, the median choice rule is also a particular case.

Since single-crossing preferences are not necessarily single-peaked (see, for instance, Example 2 in the text), this result has the important implication that the violation of single-peakedness does not preclude the existence of non-manipulable social choice functions over the real line.

Moreover, single-crossing not only may fail to satisfy single-peakedness, but also it implies that individual preferences may be correlated. Therefore, Proposition 1 also proves that the absence of independent individual preference domains is not an impediment either to find strategy-proof rules. At least for some non-trivial and common decision rules, the existence of a linear ordering of the types of the agents (with the requirement stated in Definition 1) is a sufficient condition that ensures non-manipulation at the individual level. Furthermore, as the following proposition shows, it turns out that it also guarantees non-manipulation at group level:

Proposition 2 *The median choice rule f^m is group-strategy-proof over $SC(\tilde{X})$, for any $\tilde{X} \in A(X)$.*

PROOF Consider a profile $\tilde{\succ} = (\tilde{\succ}_1, \ldots, \tilde{\succ}_n) \in SC(\tilde{X})$, with associated types $(\theta_1, \ldots, \theta_n)$. Suppose there exists a coalition $D \subseteq I$ and a list of alternative types for members of D, $(\hat{\theta}_i)_{i \in D}$, $(\hat{\theta}_i)_{i \in D} \neq (\theta_i)_{i \in D}$, such that the joint declaration generated by $\hat{\theta}_D$, $\hat{\succ}_D = (\tilde{\Phi}(\hat{\theta}_i))_{i \in D}$, produces a preferred social outcome for every member of the coalition. That is, for all $i \in D$,

$$f^m(\hat{\succ}_D, \tilde{\succ}_{D^c}) \tilde{\succ}_i f^m(\tilde{\succ}_D, \tilde{\succ}_{D^c}),$$

where $(\hat{\succ}_D, \tilde{\succ}_{D^c}) \in SC(\tilde{X})$. For simplicity, call $f^m(\tilde{\succ}) = \tilde{\tau}$ and $f^m(\hat{\succ}_D, \tilde{\succ}_{D^c}) = \hat{\tau}$. Notice that, by the definition of f^m, $\tilde{\tau}$ and $\hat{\tau}$ coincide with the tops corresponding to the orderings reported by some voters. Denote these agents j and j' and their types θ_j and $\theta_{j'}$, respectively. Since $\tilde{\tau} \neq \hat{\tau}$ assume that $\tilde{\tau} < \hat{\tau}$. Then, for all $i \in D$, $\tau(\tilde{\succ}_i) > \tilde{\tau}$. Suppose not. That is, assume $\tau(\tilde{\succ}_i) \leq \tilde{\tau}$ for some agent i in D. If $\tau(\tilde{\succ}_i) = \tilde{\tau}$, then $\tilde{\tau} \tilde{\succ}_i \hat{\tau}$, which contradicts our hypothesis. Consider, instead, that $\tau(\tilde{\succ}_i) < \tilde{\tau}$. Since $\hat{\tau} \tilde{\succ}_i \tilde{\tau}$, by single-crossing we have that for all $\theta > \theta_i$, $\hat{\tau}\, \tilde{\Phi}(\theta)\, \tilde{\tau}$. Then, θ_j has to verify that $\theta_j < \theta_i$ and, by single-crossing, $\tilde{\tau}\, \tilde{\Phi}(\theta_j)\, \tau(\tilde{\succ}_i)$ implies $\tilde{\tau}\, \tilde{\Phi}(\theta_i)\, \tau(\tilde{\succ}_i)$. Contradiction. Then, $\tau(\tilde{\succ}_i) > \tilde{\tau}$, for all $i \in D$. The rest of the proof is as follows. By definition,

$$f^m(\tilde{\succ}_D, \tilde{\succ}_{D^c}) = m^n(\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n)) = \tilde{\tau},$$

while

$$f^m(\hat{\succ}_D, \tilde{\succ}_{D^c}) = m^n(\{\tau(\hat{\succ}_i)\}_{i \in D}, \{\tau(\tilde{\succ}_j)\}_{j \in D^c}) = \hat{\tau}.$$

Two cases are possible:

1. For each $i \in D$, $\tau(\hat{\succ}_i) > \tilde{\tau}$. Then $\hat{\tau} = \tilde{\tau}$. Contradiction.

2. For some $i \in D$, $\tau(\hat{\succ}_i) \leq \tilde{\tau}$. Then, by rewritten $(\{\tau(\hat{\succ}_i)\}_{i \in D}, \{\tau(\tilde{\succ}_j)\}_{j \in D^c})$ as $(y_1, \ldots, y_n)$, we have that

$$\Big| \{j \in \{1, \ldots, n\} : y_j \leq \tilde{\tau}\} \Big| \geq \frac{(n+1)}{2}.$$

   But this implies that $m^n(y_1, \ldots, y_n) \leq \tilde{\tau}$. That is, $f(\hat{\succ}_D, \tilde{\succ}_{D^c}) \leq f(\tilde{\succ}_D, \tilde{\succ}_{D^c})$, which contradicts our initial hypothesis. □

Next we will use these positive results for the median choice rule to provide the game-theoretic counterpart of the Representative Voter Theorem. To do that, notice first that, according to the Revelation Principle, if a social choice function is *truthfully* implementable in a dominant strategy equilibrium, it must be strategy-proof. That is, strategy-proofness is a *necessary* condition for truthfully or *direct* implementation.

However, it is not *sufficient*. It is in fact sufficient when the preference domain of the social choice function can be written as a Cartesian product (Moore, 1992). Otherwise, the direct revelation mechanism is not well-defined, in the sense that the set of strategies of each agent, i.e., the set of admissible individual preference orderings that can be declared, depends on the strategies used by the others.[13]

This is precisely our case. Proposition 1 shows that f^m is strategy-proof over $SC(\tilde{X})$, for any $\tilde{X} \in A(X)$. Thus, the necessary condition for the application of the Revelation Principle holds. However, under single-crossing, individual preferences may be correlated. Therefore, $SC(\tilde{X})$ cannot be written as a Cartesian product subset of $P(\tilde{X})^n$. That is, the sufficient condition fails, and the implementation of f^m in dominant strategy equilibria has to be explicitly analyzed.

In what follows, we will informally present an extensive game form that can be used to *indirectly* implement f^m in dominant strategies. After that, we will argue that this game form is essentially equivalent to a *reduced* mechanism in normal form, and we prove that this last mechanism succeeds in implementing the median rule. We will also briefly discuss why the extensive game form or its associated reduced game form works, but not the direct mechanism in which each individual simply declares his top in $\tilde{X}$. Finally, we will derive the game-theoretic equivalent of Theorem 1.

2.5.1 Implementation of the median choice rule

Suppose individuals in I have preferences $(\succ_1, \ldots, \succ_n) \in SC(X)$. Assume the selection of a social outcome in $\tilde{X}$, which is the planner's basic problem, is indirectly performed by the following two-stage voting procedure. In the first stage, individuals select by pairwise majority voting a *representative* individual from the set I. Then, in the second stage, the winner chooses an alternative in $\tilde{X}$, which is then the policy implemented by the planner.

[13]A possible way of solving this consists in asking to each individual to report a preference profile, instead of his individual preference ordering. If the social choice function is strategy-proof, then it can be shown that reporting the true preferences of the whole society is a dominant strategy for each individual. See Osborne and Rubinstein (1994) for a formal proof.

Since in the last stage each individual i has a dominant strategy, which is simply to choose his most preferred alternative in $\tilde{X}$, $\tau(\tilde{\succ}_i)$, it is immediate to see that this extensive game form is equivalent to a *reduced* strategic game form in which individuals choose by pairwise majority comparisons an alternative in the set $T(\tilde{X}, \tilde{\succ}) = \{\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_i), \ldots, \tau(\tilde{\succ}_n)\}$.

Now we prove that this reduced mechanism can be used to implement f^m in a dominant strategy equilibrium.

Definition 6 *A mechanism Γ with consequences in $\tilde{X}$ is a strategic game form $\langle I, (S_i), \phi\rangle$ where, for each $i \in I$, S_i is the set of actions available for agent i, and $\phi : \prod_{i\in I} S_i \to \tilde{X}$ is an outcome function that associates an alternative with every action profile.*

We say that Γ implements a social choice function $f : SC(\tilde{X}) \to \tilde{X}$ in dominant strategies if there exists a dominant strategy equilibrium for the mechanism, yielding the same outcome as f for each possible preference profile $\tilde{\succ} \in SC(\tilde{X})$. This is formally stated in Definition 7.

Definition 7 *The mechanism $\Gamma = \langle I, (S_i), \phi\rangle$ implements the social choice function $f : SC(\tilde{X}) \to \tilde{X}$ in dominant strategies if there exists a dominant strategy equilibrium of Γ, $s^*(\cdot) = (s_1^*(\cdot), \ldots, s_n^*(\cdot))$, such that $\phi(s^*(\tilde{\succ})) = f(\tilde{\succ})$ for all $\tilde{\succ} \in SC(\tilde{X})$.*

Proposition 3 *There exists a mechanism that implements $f^m : SC(\tilde{X}) \to \tilde{X}$ in dominant strategies over $\tilde{X}$.*

Proof Consider a preference profile $\tilde{\succ} \in SC(\tilde{X})$ and the mechanism $\Gamma = \langle I, (S_i), \phi\rangle$, where I is the set of players; an action for agent $i \in I$ is simply to choose an element in $S_i = T(\tilde{X}, \tilde{\succ}) = \{\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_i), \ldots, \tau(\tilde{\succ}_n)\}$; and the outcome function $\phi(s_1, \ldots, s_n) = m^n(s_1, \ldots, s_n)$. We will show that the action profile $(\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n))$ constitutes a dominant strategy equilibrium of the game induced by Γ. That is,

$$\phi(s_1, \ldots, \tau(\tilde{\succ}_i), \ldots, s_n) \;\tilde{\succ}_i\; \phi(s_1, \ldots, \hat{s}_i, \ldots, s_n)$$

for all i, $\hat{s}_i \neq \tau(\tilde{\succ}_i)$, $s_{-i} \in \prod_{j\neq i} S_j$. Since, by definition, $\phi(\cdot) = m^n(\cdot)$, we can easily recast the proof of Proposition 1 to fit in this scheme. Suppose that there exists such $\hat{s}_i$. Call $\tilde{s} = \phi(\tau(\tilde{\succ}_i), s_{-i})$ and $\hat{s} = \phi(\hat{s}_i, s_{-i})$. Without loss of generality, assume $\tau(\tilde{\succ}_i) < \tilde{s}$. We have two cases to consider:

1. $\hat{s}_i \leq \tilde{s}$. Then, $m^n(\tau(\tilde{\succ}_i), s_{-i}) = m^n(\hat{s}_i, s_{-i})$ and, therefore, $\phi(\tau(\tilde{\succ}_i), s_{-i}) = \phi(\hat{s}_i, s_{-i})$. Contradiction.

2. $\hat{s}_i > \tilde{s}$. Then the new median $\hat{s}$ will be in the interval $[\tilde{s}, \hat{s}_i]$. By hypothesis, $\hat{s} \tilde{\succ}_i \tilde{s}$. Furthermore, since the preferences are single-crossing on $T(\tilde{X}, \tilde{\succ})$ and $\hat{s} > \tilde{s}$, for every $\theta > \theta_i$ we have that $\hat{s}\, \tilde{\Phi}(\theta)\, \tilde{s}$. On the other hand, notice that, since each $S_j = T(\tilde{X}, \tilde{\succ})$, there must exist $\theta_j \in \Theta$ such that $\tilde{s} = \tau(\tilde{\Phi}(\theta_j))$. Moreover, θ_j must be such that $\theta_j < \theta_i$. But then, since $\tau(\tilde{\succ}_i) < \tilde{s}$ and $\tilde{s}\, \tilde{\Phi}(\theta_j)\, \tau(\tilde{\succ}_i)$, by single-crossing, we have that $\tilde{s}\, \tilde{\Phi}(\theta)\, \tau(\tilde{\succ}_i)$ for all $\theta > \theta_j$; in particular for θ_i. Contradiction.

Therefore, $(\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n))$ is a dominant strategy equilibrium. □

The fact that the alternative declared by each agent is restricted to belong to $T(\tilde{X}, \tilde{\succ})$, the set of all individual maximal alternatives in $\tilde{X}$, is crucial for the proof of Proposition 3. It is easy to see that a mechanism based on direct declarations of the most preferred alternatives in $\tilde{X}$ cannot be used to implement f^m. For instance, in Example 2, if agents are asked to declare their most preferred alternatives in $\tilde{X} = \{x, y, z\}$, then manipulation cannot be avoided: if agent 1 and agent 3 declare y and z, respectively, then player 2 will prefer to announce z instead of his true top x.[14]

Instead, the reason of why our *indirect* mechanism works is because the induced preferences over the set $T(\tilde{X}, \tilde{\succ})$, derived from $\tilde{\succ} \in SC(\tilde{X})$, are single-peaked. This is formally shown in Lemma 3 below.[15]

Lemma 3 *If a preference profile $\tilde{\succ} = (\tilde{\succ}_1, \ldots, \tilde{\succ}_n)$ is single-crossing over $\tilde{X}$, then the restriction of $\tilde{\succ}$ over the set $T(\tilde{X}, \tilde{\succ})$ is single-peaked.*

Proof For a given profile $(\tilde{\succ}_1, \ldots, \tilde{\succ}_n) \in SC(\tilde{X})$ and the associated set $T(\tilde{X}, \tilde{\succ})$, consider the restriction of $\tilde{\succ}$ to $T(\tilde{X}, \tilde{\succ})$, denoted $\tilde{\succ}^T = (\tilde{\succ}^T_1, \ldots, \tilde{\succ}^T_n)$. By contradiction, suppose $\tilde{\succ}^T \notin SP(T)^n$, where $SP(T)^n$ is the set of all single-peaked preference profiles over $T(\tilde{X}, \tilde{\succ})$ (with respect to the linear order $\leq$). Then, there exist an individual $i \in I$, with type $\theta_i \in \Theta$, and x, y, $\tau(\tilde{\succ}_i) \in T(\tilde{X}, \tilde{\succ})$ such that

$$x < y \leq \tau(\tilde{\succ}_i), \text{ but } x \tilde{\succ}^T_i y.$$

Thus, $y \neq \tau(\tilde{\succ}_i)$. Moreover, since $\tilde{\succ}^T \in SC(T)$, $x \tilde{\succ}^T_j y$ for all $\theta_j \leq \theta_i$. This means $y \neq \tau(\tilde{\succ}_j)$ for all $\theta_j \in \{\theta_1, \theta_2, \ldots, \theta_i\}$. However, since we

[14] Proposition 1 shows that individual manipulation is ruled out when agents are required to declare a complete preference ordering, and not just the top alternative. The intuition is again illustrated by Example 2. Notice that in this case individual 1 cannot summit an ordering with the alternative y as its top without violating the single-crossing condition. Thus, player 2 has no reason to lie.

[15] Notice that $T(\tilde{X}, \tilde{\succ})$ can be identified with the set of *actual* ideal points, since $T(\tilde{X}, \tilde{\succ}) = \{x \in \tilde{X} : \exists i \in I \text{ such that } x = \tau(\tilde{\succ}_i)\}$.

assume $y \in T(\tilde{X}, \tilde{\succ})$, then $y = \tau(\tilde{\succ}_k)$ for some individual $k \in I$ with type $\theta_k \in \{\theta_{i+1}, \theta_{i+2}, \ldots, \theta_n\}$. Then, $y \,\tilde{\succ}_k\, \tau(\tilde{\succ}_i)$ implies $y \,\tilde{\succ}_j\, \tau(\tilde{\succ}_i)$ for all $\theta_j \leq \theta_k$. In particular, for θ_i. Contradiction. The same argument applies if $\tau(\tilde{\succ}_i) \leq y < x$ and $x \,\tilde{\succ}_i^T\, y$. Hence, $\tilde{\succ}^T \in SP(T)^n$. □

It is easy to show that the converse of Lemma 3 does not hold. That is, preferences can be single-peaked over $T(\tilde{X}, \tilde{\succ})$, but not necessarily single-crossing on $T(\tilde{X}, \tilde{\succ})$. The preference profile presented in Table 2.4 below provides an example in which this happens.

Table 2.4: Counterexample

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$
w	x	y	z
x	y	x	y
y	z	w	x
z	w	z	w

Finally, we derive the following Corollary from Proposition 3:

Corollary 1 *For any $\tilde{X} \in A(X)$, there exists a mechanism that implements $f^m : OR(\tilde{X}) \to \tilde{X}$ in dominant strategies over $\tilde{X}$.*

Proof Trivial. Consider any preference profile $\tilde{\succ} \in OR(\tilde{X})$. By Lemma 2, there exists a permutation $\bar{\gamma}$ of Θ that generates a profile $\tilde{\succ}^{\bar{\gamma}} \in SC(\tilde{X})$. Hence, the mechanism defined in Proposition 3 yields, as the outcome of its dominant strategy equilibrium, the median value of the maximal alternatives over $\tilde{X}$, $\phi(\tilde{\succ}^{\bar{\gamma}}) = m^n(\tau(\tilde{\succ}_1^{\bar{\gamma}}), \ldots, \tau(\tilde{\succ}_n^{\bar{\gamma}})) = \tau(\tilde{\Phi}(\theta_r^{\bar{\gamma}}))$. Finally, this outcome coincides with $f^m(\tilde{\succ})$ because, as seen in Theorem 1, $m^n(\theta_1^{\bar{\gamma}}, \ldots, \theta_n^{\bar{\gamma}}) = m^n(\theta_1, \ldots, \theta_n)$. □

This Corollary provides the *strategic counterpart* of Theorem 1. That is, it shows that, when preferences are order-restricted, the social outcome under pairwise majority voting, i.e. the most preferred alternative of the median type, can be attained by a reduced mechanism in which agents are allowed to declare one of the individual maximal alternatives in the feasible set of policies. Or, alternatively, it can be achieved by following a two-stage voting procedure in which, first, the individuals select a representative among themselves, and then the representative voter chooses a policy to be implemented by the planner.

2.6 Final remarks

In this chapter, we exhibited several results. First of all, we have proven that, apart from single-peakedness, there exists another very *natural* preference domain over the real line for which strategy-proof choice rules can be found. Concretely, we have shown that single-crossing preferences constitute a domain restriction that allows not only majority voting equilibria, but also the existence of non-trivial strategy-proof (as well as group-strategy-proof) social choice functions. In particular, this is true for the median choice rule.

The first feature to remark of this result is that single-crossing preferences do not necessarily satisfy single-peakedness. But, as it is known, in one-dimensional collective decision models this is one of the most frequently applied domain restrictions that guarantee strategy-proofness. Thus, the result found here shows that the violation of single-peakedness does not preclude the existence of non-manipulable social choice functions over the real line.

Furthermore, single-crossing also implies that individual preferences are correlated. Therefore, Proposition 1 also proves that the absence of independent individual preference domains is not an obstacle for the existence of strategy-proof rules. At least for some non-trivial and common decision rules, the existence of a certain kind of linear ordering of the types of the agents is a sufficient condition that ensures non-manipulation both at the individual and at the group level.

Another important results are summarized in Lemmas 1 and 2, which exhibit the close relation between single-crossing and order-restriction. A previous work in the same direction is Gans and Smart (1996), in which these preference domains are shown to be essentially equivalent. Nevertheless, our results differ from theirs in two ways. First, ours seem to be more consistent with Rothstein's original characterization of order-restriction. Second, particular attention is devoted here to the fact that these conditions may not be *directly* equivalent. The crucial point to understand this difference is that, unlike single-crossing, order-restriction does not assumes any ordering on the set of possible alternatives. Furthermore, it is precisely this feature that makes order-restriction so interesting for analyzing strategy-proofness in multi-dimensional choice spaces and over restricted agendas.

Finally, these previous results are used at the end of the chapter to show that the Representative Voter Theorem has a well-defined non-cooperative strategic foundation. Concretely, we show that the collective outcome predicted by this Theorem can be implemented through a simple sequential mechanism in which, first, individuals select a representative among themselves, and then the representative voter chooses a policy to be implemented by the planner. Given that the structure of this mechanism presents some features that we observe frequently in "real" voting processes, the analysis carried out here may also provide insights for a rationale of these "real" voting situations.

At the same time, there are significant topics that this chapter does not cover. The most important task that we have left for future work is to fully characterize the family of strategy-proof social choice functions over single-crossing preference profiles. Of course, the classes that also satisfy other requirements like anonymity, Pareto efficiency or combinations of them should also be determined.

The second relevant aspect that we do not address here is how these results change when individual preference orderings are allowed to express indifference between different alternatives. Clearly, our simplification is justified by the fact that the set of possible social outcomes is finite. However, we guess substantial changes may be expected in our results if this assumption is dropped.

Finally, another problem that must be answered is how to extend single-crossing and order-restriction to multidimensional spaces. That is, we should consider the way in which these preference restrictions can deal with both multidimensional choice sets and political conflicts of interests that cannot be projected onto a one-dimensional space.

Chapter 3

On strategy-proofness and single-crossing

3.1 Introduction

It is well-known both in modern economic theory and positive political science that voting, in general, can fail to produce well-defined collective outcomes. For instance, the conflict of interests in a society may be such that none of the feasible social alternatives has the support of a majority of voters against any other alternative. Furthermore, it is also known that none of the aggregation methods via voting are free of individual and group manipulation.

To overcome these negative results, it is common in social choice theory to place restrictions on individual preferences. This allows to study the properties of these voting procedures by looking at more homogenous societies. If the social alternatives can be placed over the real line, as for instance when different levels of a public good or different tax rates are the subject of collective choice, one of the most common preference restrictions is *single-crossing*.[1]

This restriction makes sense in many political settings. In few words, a society has single-crossing preferences if, given any two policies, one of them more to the right than the other, the more rightist is an individual (with respect to another individual) the more he will prefer the right-wing policy over the left-wing one. For instance, if alternatives are tax rates and individuals are *ordered* according to their income, this restriction means simply that, the richer is an individual the lower will be the tax rate he will prefer.

[1] The other one is, of course, single-peakedness.

Technically, this condition not only guarantees the existence of majority voting equilibria, but it also provides a simple characterization of the core of the majority rule. In fact, the core is simply the set of ideal points of the median agent in the ordering of the individuals that makes the preference profile single-crossing. This result is sometimes referred to as the *Representative Voter Theorem* (Rothstein, 1991) or, alternatively, as "the second version" of the *Median Voter Theorem* (Myerson, 1996 and Gans and Smart, 1996).

In any case, the main problem with this result is that, unlike the original Median Voter Theorem over single-peaked preferences, whose non-cooperative foundation was provided by Black (1948), first, and then by Moulin (1980), the Representative Voter Theorem is based on the assumption that individuals honestly reveal their preferences. That is, it is derived assuming *sincere* voting. In effect, even though single-crossing is now largely used in models of collective decision-making, nothing has been said in the literature about the possibility of manipulation (strategic voting) over this preference domain. Moreover, the "single-crossing version" of the Median Voter Theorem is usually applied without caring much about its strategic foundations.

This issue has been considered in the last chapter. It has been shown there that the single-crossing condition guarantees not only majority voting equilibria, but also non-manipulable choice rules. In particular, it showed that this is true for the median choice rule, which is found to be strategy-proof as well as group-strategic-proof. As a by-product, it has also proved that the collective outcome predicted by the Representative Voter Theorem can be implemented in dominant strategies through a simple mechanism. This mechanism is a two-stage voting procedure in which, first, individuals select a representative among themselves, and then the representative voter chooses a policy to be implemented by the planner.

Taken this as a starting point, this chapter characterizes the whole family of strategy-proof social choice functions over the domain of single-crossing preference profiles. The main result shows that this family is completely described by the class of *positional dictator* choice rules; i.e. by all those rules derived from the extended median rule by distributing phantom voters at the extremes of the extended non-negative real line. This class is shown to be strategy-proof as well as group-strategy-proof. Moreover, it is also proved that those rules are non-manipulable not only over the full set of alternatives, but also over every possible policy *agenda*. Interestingly, the chapter shows that, for this kind of individual preferences, the above results cannot be extended to other median voter schemes.

3.2 The model, notation and definitions

The basic model of single-crossing preferences assumes that the set of agents I is finite and its cardinality $|I| = n > 2$ is odd. Individuals in I must choose a policy (for example, the level of a given local tax) from a feasible set of social alternatives. They do this by voting.

The set of all possible collective outcomes $X = \{x_1, \ldots, x_l\}$, $|X| > 2$, is assumed to be a finite subset of the extended non-negative real line $\mathbf{R}_+^* = \Re_+ \cup \{+\infty\}$. The set X is such that $x_j \leq x_k$ for $j \leq k$, where the linear order $\leq$ is the usual order on $\mathbf{R}_+^*$. For a vector $x = (x_1, \ldots, x_n) \in (\mathbf{R}_+^*)^n$, we let $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ and $(\hat{x}_i, x_{-i}) = (x_1, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_n)$, where $\hat{x}_i \in \mathbf{R}_+^*$. In addition, for any group of agents $D \subseteq I$, we denote $(x_D, x_{D^c}) = ((x_i)_{i \in D}, (x_j)_{j \in D^c})$, where $D^c = I \backslash D$.

The set of all feasible alternatives may be either the entire X or just one of its non-empty subsets. The set $\tilde{X}$ represents a generic subset - with the induced order - of X. We use $A(X)$ to represent the set of all non-empty subsets of X, $A(X) = \{\tilde{X} : \tilde{X} \in 2^X \backslash \emptyset\}$. In words, X is the universal set of outcomes, whereas a particular situation, or *agenda*, involves a $\tilde{X} \in A(X)$.

Let $P(X)$ be the set of all complete, transitive and antisymmetric binary orderings of X. We say $P(X)$ is the *universal domain* of individual preferences.[2] Agent i's preferences over the alternatives in X are assumed to be completely characterized by a single parameter $\theta_i \in \Theta = \{\theta^1, \ldots, \theta^m\}$, where $\Theta \subset \Re$ is a finite and *ordered* subset of the real line, such that $\theta^1 < \theta^2 < \ldots < \theta^m$ and $m \leq |P(X)|$. As usual, we interpret θ_i as being agent i's *type*.

That is, we assume there exists a function $\Phi : \Theta \to P(X)$ that assigns a unique element $\succ_\theta \in P(X)$ to each $\theta \in \Theta$. We say that $\succ_i$ represents the preferences of an agent i of type θ_i if,

$$\forall x, y \in X, \; x \succ_i y \; \Leftrightarrow \; x \, \Phi(\theta_i) \, y.$$

The maximal set associated with the pair $\langle X, \succ_i \rangle$ is $M(X, \succ_i) = \{x \in X : \forall y \in X \backslash \{x\}, x \succ_i y\}$. That is, $M(X, \succ_i)$ yields the alternative that is top-ranked in X for i with respect to her preferences $\succ_i$. Notice that since preferences are strict, maximal sets are indeed singletons.

A preference profile associated to a profile of types $\theta = (\theta_1, \ldots, \theta_n) \in \Theta^n$ is an n-tuple $(\succ_1, \ldots, \succ_n) = (\Phi(\theta_1), \ldots, \Phi(\theta_n))$ in $P(X)^n$. This means that the profile of individual preferences depends on the *state* $\theta \in \Theta^n$: in the state θ, agent i has preferences $\Phi(\theta_i)$ over the set X. This formulation

[2]Indifference between alternatives is not allowed. This is a natural assumption when the set of alternatives is finite.

allows for any degree of correlation across the agents' preferences. We assume each agent observes θ, so that there exists complete information among the agents about their preferences over X. Extending our earlier conventions to preference profiles, we have that $\succ_{-i} = (\succ_1, \ldots, \succ_{i-1}, \succ_{i+1}, \ldots, \succ_n)$. Similarly, the profile obtained by changing agent i's preferences for $\hat{\succ}_i$ is $(\hat{\succ}_i, \succ_{-i}) = (\succ_1, \ldots, \succ_{i-1}, \hat{\succ}_i, \succ_{i+1}, \ldots, \succ_n)$. Finally, for any group of agents $D \subseteq I$, $(\succ_D, \succ_{D^c}) = ((\succ_i)_{i \in D}, (\succ_j)_{j \in D^c})$.

Now, we restrict the set of admissible preference profiles by imposing a condition on preferences that involves the entire profile:

Definition 1 *A preference profile $(\succ_1, \ldots, \succ_n)$ derived from $\Phi : \Theta \to P(X)$ is single-crossing on X if, for all x, $y \in X$ and all i, $j \in I$ such that either $y > x$ and $\theta_j > \theta_i$ or $y < x$ and $\theta_j < \theta_i$,*

$$y \, \Phi(\theta_i) \, x \quad \Rightarrow \quad y \, \Phi(\theta_j) \, x.$$

We denote $SC(X)$ the set of all single-crossing preference profiles on X.[3] The recent interest on this restricted domain of preferences is due to the fact that, like single-peakedness,[4] single-crossing has been shown to be sufficient to guarantee the existence of majority voting equilibria. However, apart from this fact, it should be clear that both domain conditions are independent, in the sense that neither property is logically implied by the other. Tables 3.1, 3.2 and 3.3 below show three situations (for the simplest possible case of three individuals and three alternatives) in which preferences are, respectively, (1) single-crossing, but not single-peaked; (2) single-peaked, but not single-crossing; and (3) both single-crossing and single-peaked.

In the political arena, single-crossing makes sense in many applications. For instance, suppose individual types are interpreted as being different ideological characters, arranged in the left-right scale, and the alternatives as public policies to be chosen by the society. Then, preferences are single-crossing if, for any two policies, one of them more to the right than the other, the more rightist is a type, the more will he prefer the right-wing policy over the left-wing one.

[3]Other expressions used in the literature to denominate similar preference restrictions are *hierarchical adherence*, *order-restriction* and *unidimensional alignment*. For more on them, see Roberts (1977), Rothstein (1990, 1991), Gans and Smart (1996), Austen-Smith and Banks (1999) and List (2001), and the references quoted there.

[4]Formally, a preference profile $(\succ_1, \ldots, \succ_n)$ is single-peaked on X with respect to the linear order $\leq$ if for all $i \in I$, there exists $\tau_i \in X$, called the *peak* of i associated to the preference relation $\succ_i$, such that (1) $\tau_i \succ_i x$, for all $x \in X \setminus \{\tau_i\}$; (2) $y < x \leq \tau_i$ implies $x \succ_i y$, and (3) $\tau_i \leq x < y$ implies $x \succ_i y$.

Table 3.1: Single-crossing

$\Phi(\theta^1)$	$\Phi(\theta^2)$	$\Phi(\theta^3)$
x_1	x_1	x_3
x_2	x_3	x_2
x_3	x_2	x_1

Table 3.2: Single-peakedness

$\succ_1$	$\succ_2$	$\succ_3$
x_1	x_4	x_2
x_2	x_2	x_1
x_4	x_1	x_3
x_3	x_3	x_4

Given a preference $\succ_i$ in the profile $\succ \in SC(X)$, we define agent i's *induced* preferences over the agenda $\tilde{X} \in A(X)$, $\tilde{\succ}_i$, as follows:

$$\forall x,\, y \in \tilde{X},\; x \,\tilde{\succ}_i\, y \;\Leftrightarrow\; x \succ_i y.$$

Notice that the property of being single-crossing is preserved in the induced preferences. That is, if $\succ \in SC(X)$ then $\tilde{\succ} \in SC(\tilde{X})$, for all $\tilde{X} \in A(X)$.

These preferences can be aggregated. The input for this aggregation process is the set of *declarations* of the individuals. These declarations are intended to provide information about their true types, although their sincerity cannot be ensured.

The aggregation process is represented by a social choice function. For any $\tilde{X} \in A(X)$, a *social choice function* f on $SC(\tilde{X})$ is a single-value mapping $f : SC(\tilde{X}) \to \tilde{X}$ that associates to each preference profile $\tilde{\succ} = (\tilde{\succ}_1, \ldots, \tilde{\succ}_n) \in SC(\tilde{X})$ a unique outcome $f(\tilde{\succ}) \in \tilde{X}$.

We will be interested in social choice functions that satisfy the following properties. The main one is that agents, acting individually or in groups, never have the incentives to misrepresent their preferences. To capture this idea, we define the following two concepts:

Definition 2 *A social choice function f on $SC(\tilde{X})$ is strategy-proof if for all $\tilde{\succ} \in SC(\tilde{X})$, and for any agent $i \in I$, with type θ_i, any misrepre-*

Table 3.3: Single-crossing and single-peakedness

$\Phi(\theta^1)$	$\Phi(\theta^2)$	$\Phi(\theta^3)$
x_1	x_2	x_3
x_2	x_1	x_2
x_3	x_3	x_1

sentation $\hat{\succ}_i = \tilde{\Phi}(\hat{\theta}_i)$, $\hat{\theta}_i \neq \theta_i$, *is such that either* $f(\tilde{\succ}) \,\tilde{\succ}_i\, f(\hat{\succ}_i, \tilde{\succ}_{-i})$ *or* $f(\tilde{\succ}) = f(\hat{\succ}_i, \tilde{\succ}_{-i})$, *where* $(\hat{\succ}_i, \tilde{\succ}_{-i}) \in SC(\tilde{X})$.[5]

If a social choice function f is not strategy-proof, then there exist $i \in I$ and $\hat{\succ}_i$ such that for some $\tilde{\succ}_{-i}$, $(\hat{\succ}_i, \tilde{\succ}_{-i}) \in SC(\tilde{X})$, and i's true preferences, $\tilde{\succ}_i$, $f(\hat{\succ}_i, \tilde{\succ}_{-i}) \,\tilde{\succ}_i\, f(\tilde{\succ}_i, \tilde{\succ}_{-i})$. Then, we say f is *manipulable* at $(\tilde{\succ}_i, \tilde{\succ}_{-i})$, by i, via $\hat{\succ}_i$. In the same way:

Definition 3 *A social choice function* f *on* $SC(\tilde{X})$ *is group-strategy-proof if for all* $\tilde{\succ} \in SC(\tilde{X})$, *and for every coalition* $D \subseteq I$, *with types* $\theta_D = (\theta_i)_{i\in D}$, *there does not exist a joint misrepresentation* $\hat{\succ}_D = (\tilde{\Phi}(\hat{\theta}_i))_{i\in D}$, $\hat{\theta}_D \neq \theta_D$, *such that, for all* $i \in D$, $f(\hat{\succ}_D, \tilde{\succ}_{D^c}) \,\tilde{\succ}_i\, f(\tilde{\succ})$, *where* $(\hat{\succ}_D, \tilde{\succ}_{D^c}) \in SC(\tilde{X})$.

Another crucial property we may seek in a social choice function is *Pareto efficiency*. This condition is well-known and requires no further comment here:

Definition 4 *A social choice function* f *on* $SC(\tilde{X})$ *is Pareto efficient if and only if, for all* $\tilde{\succ} \in SC(\tilde{X})$,

$$f(\tilde{\succ}) \in \{x \in \tilde{X} : \nexists\, y \in \tilde{X} \text{ such that } y \,\tilde{\succ}_i\, x \;\forall i \in I\}.$$

One last property a social choice function may satisfy is *tops-onliness.* We say that f is *tops-only* if for any profile of preferences $\tilde{\succ} \in SC(\tilde{X})$ the social outcome $f(\tilde{\succ})$ is determined only by the individuals' most-preferred alternatives in $\tilde{\succ}$. Formally, for any individual ordering $\tilde{\succ}_i$ in $\tilde{\succ} \in SC(\tilde{X})$, let $\tau(\tilde{\succ}_i) = M(\tilde{X}, \tilde{\succ}_i)$:

Definition 5 *A social choice function* f *on* $SC(\tilde{X})$ *is tops-only if for any two preference profiles* $\tilde{\succ}$ *and* $\hat{\succ}$ *in* $SC(\tilde{X})$, *such that for any* $i \in I$, $\tau(\tilde{\succ}_i) = \tau(\hat{\succ}_i)$, $f(\tilde{\succ}) = f(\hat{\succ})$.

[5]With $\tilde{\Phi}(\hat{\theta}_i)$ we represent the restriction of $\Phi(\hat{\theta}_i)$ over $\tilde{X}$.

Of course, the tops-only property dramatically constraints the scope for manipulation: no agent can expect to be able to affect the social outcome without modifying the peak of his reported preference ordering. However, as we will show, this condition is related to the strategy-proofness condition itself. In effect, when preferences are single-crossing, it turns out that every strategy-proof social choice rule whose range is greater than two must be tops-only (see Corollary 3 below).

We now define the *extended median rule*. This social choice function is a particular member of the class of *anonymous* and tops-only choice rules,[6] which provides a natural extension of the basic idea of the *median choice rule*.

For any odd positive integer k, let $m^k : (\mathbf{R}_+^*)^k \rightarrow \mathbf{R}_+^*$ be the *k-median function*, defined in the following way: for all $x \in (\mathbf{R}_+^*)^k$, $m^k(x)$ is the *k-median* of $x = (x_1, \ldots, x_k)$ if and only if $|\{x_i \in \mathbf{R}_+^* : x_i \leq m^k(x)\}| \geq \frac{(k+1)}{2}$ and $|\{x_j \in \mathbf{R}_+^* : m^k(x) \leq x_j\}| \geq \frac{(k+1)}{2}$. Because k is odd, this function is always well-defined. Now, we define the *extended median rule* in the following way:

Definition 6 *A social choice function f^e on $SC(\tilde{X})$ is called the extended median rule if there exist $n+1$ real numbers $\alpha_1, \ldots, \alpha_{n+1} \in \mathbf{R}_+^*$, called the* phantom voters*, such that, for all $\tilde{\succ} \in SC(\tilde{X})$,*

$$f^e(\tilde{\succ}) = m^{2n+1}(\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n), \alpha_1, \ldots, \alpha_{n+1}).$$

A particular case of this rule is the following. Let $\alpha_1 = \ldots = \alpha_{\frac{n+1}{2}} = 0$ and $\alpha_{\frac{n+1}{2}+1} = \ldots = \alpha_{n+1} = +\infty$. Then,

$$f^e(\tilde{\succ}) = m^{2n+1}(\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n), \underbrace{0, \ldots, 0}_{\frac{(n+1)}{2}\text{ times}}, \underbrace{+\infty, \ldots, +\infty}_{\frac{(n+1)}{2}\text{ times}}),$$

is the well-known *median choice rule*, f^m, that can be re-written as

$$f^m(\tilde{\succ}) = m^n(\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n)).$$

Proceeding in the same way, other *supermajority* rules can also be derived from f^e, by restricting the parameters $\alpha_1, \ldots, \alpha_{n+1}$ to take some particular values in $\mathbf{R}_+^*$. Notice that, if $\alpha_1 =, \ldots, = \alpha_{n+1} = \alpha$, f^e is completely insensitive to the preferences reported by the individuals, since $\forall \tilde{\succ} \in SC(\tilde{X})$

$$f^e(\tilde{\succ}) = m^{2n+1}(\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n), \underbrace{\alpha, \ldots, \alpha}_{(n+1)\text{ times}}) = \alpha.$$

[6] A social choice function f on $SC(\tilde{X})$ is *anonymous* if for any $\tilde{\succ}$ and $\hat{\succ}$ in $SC(\tilde{X})$, such that $\hat{\succ}$ is a permutation of $\tilde{\succ}$, $f(\tilde{\succ}) = f(\hat{\succ})$.

We might want to exclude such undesirable voting rules and, in particular, require Pareto efficiency. In order to allow the extended median rule f^e to satisfy Pareto efficiency, we eliminate the possibility of inefficiency by setting $\alpha_n = 0$ and $\alpha_{n+1} = +\infty$. Therefore, we obtain the following restriction of f^e:

$$f^{e^*}(\tilde{\succ}) = m^{2n-1}(\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n), \alpha_1, \ldots, \alpha_{n-1}),$$

which is the *efficient extended median rule* with $n-1$ parameters.

In the following section, we will study how well the extended median rule performs, according to the manipulation criteria given above, on the domain of single-crossing preference profiles.

3.3 Main results

Suppose phantom voters are restricted to having peaks at either zero or infinity. That is, assume that, for any $i = 1, \ldots, n-1$, $\alpha_i \in \{0, +\infty\}$, such that each phantom voter is either a *leftist* or a *rightist*. For this particular case in which all ficticious voters take only the extreme values on $\mathbf{R}^*_+$, the Condorcet winners obtained are the well-known class of *positional dictators*.[7]

These rules select the jth ranked peak among the tops of the reported preference orderings, for some $j = 1, \ldots, n$. For example, if $j = 1$, we have the *leftist rule*, which chooses the smallest reported peak of a real voter. Of course, the median rule is also a particular case. It turns out that all these rules are group-strategy-proof over $SC(\tilde{X})$, for any $\tilde{X} \in A(X)$:[8]

Proposition 1 *Let $\alpha_1, \ldots, \alpha_{n-1}$ be such that $\alpha_1, \ldots, \alpha_{n-1} \in \{0, +\infty\}$. Then, the extended median rule f^{e^*} is group-strategy-proof over $SC(\tilde{X})$, for any $\tilde{X} \in A(X)$.*[9]

Proof Consider a profile $\tilde{\succ} = (\tilde{\succ}_1, \ldots, \tilde{\succ}_n)$ in $SC(\tilde{X})$, with associated types $(\theta_1, \ldots, \theta_n)$. Suppose there exists a coalition $D \subseteq I$ and a list of alternative

[7]See Moulin (1988), pp. 302.

[8]To put the phantoms at some point that coincides with the peak of some actual type of the voters, in addition to at zero or infinity, yields the same results. However, we ruled out this possibility for two reasons. First, because then the phantoms and, therefore, the social choice function would depend on the particular profile of preferences considered. Second, and more important, because otherwise to define the choice rule the planner would require information about the actual location of the true tops. But this is precisely one of the problems that he tries to solve by means of the voting process.

[9]A similar result holds for f^e. That is, efficiency may be dropped without altering the result of Proposition 1.

types for members of D, $(\hat{\theta}_i)_{i\in D}$, $(\hat{\theta}_i)_{i\in D} \neq (\theta_i)_{i\in D}$, such that the joint declaration generated by $(\hat{\theta}_i)$, $\hat{\succ}_D = (\tilde{\Phi}(\hat{\theta}_i))_{i\in D}$, produces a preferred social outcome for every member of the coalition. That is, for all $i \in D$,

$$f^{e^*}(\hat{\succ}_D, \tilde{\succ}_{D^c}) \, \tilde{\succ}_i \, f^{e^*}(\tilde{\succ}_D, \tilde{\succ}_{D^c}),$$

where $(\hat{\succ}_D, \tilde{\succ}_{D^c}) \in SC(\tilde{X})$. For simplicity, call $f^{e^*}(\tilde{\succ}) = \tau$ and $f^{e^*}(\hat{\succ}_D, \tilde{\succ}_{D^c}) = \hat{\tau}$. Notice that, by the assumed distribution of the phantom voters, τ and $\hat{\tau}$ must coincide with the tops reported by some "real" voters. Denote these agents j and j' and their types θ_j and $\theta_{j'}$, respectively. Since $\tau \neq \hat{\tau}$, assume that $\tau < \hat{\tau}$. Then, for all $i \in D$, $\tau(\tilde{\succ}_i) > \tau$. Suppose not. That is, assume $\tau(\tilde{\succ}_i) \leq \tau$ for some agent i in D. If $\tau(\tilde{\succ}_i) = \tau$, then $\tau \, \tilde{\succ}_i \, \hat{\tau}$, which contradicts our hypothesis. Consider, instead, that $\tau(\tilde{\succ}_i) < \tau$. Since $\hat{\tau} \, \tilde{\succ}_i \, \tau$, by single-crossing we have that for all $\theta > \theta_i$, $\hat{\tau} \, \tilde{\Phi}(\theta) \, \tau$. Then, θ_j has to verify that $\theta_j < \theta_i$ and, by single-crossing, $\tau \, \tilde{\Phi}(\theta_j) \, \tau(\tilde{\succ}_i)$ implies $\tau \, \tilde{\Phi}(\theta_i) \, \tau(\tilde{\succ}_i)$. Contradiction. Then, $\tau(\tilde{\succ}_i) > \tau$, for all $i \in D$. The rest of the proof is as follows. By definition,

$$f^{e^*}(\tilde{\succ}_D, \tilde{\succ}_{D^c}) = m^{2n-1}(\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n), \alpha_1, \ldots, \alpha_{n-1}) = \tau,$$

while

$$f^{e^*}(\hat{\succ}_D, \tilde{\succ}_{D^c}) = m^{2n-1}(\{\tau(\hat{\succ}_i)\}_{i\in D}, \{\tau(\tilde{\succ}_j)\}_{j\in D^c}, \alpha_1, \ldots, \alpha_{n-1}) = \hat{\tau}.$$

Two cases are possible:

1. For each $i \in D$, $\tau(\hat{\succ}_i) > \tau$. Then $\hat{\tau} = \tau$. Contradiction.

2. For some $i \in D$, $\tau(\hat{\succ}_i) \leq \tau$. Then, if we rename $(\{\tau(\hat{\succ}_i)\}_{i\in D}, \{\tau(\tilde{\succ}_j)\}_{j\in D^c}, \alpha_1, \ldots, \alpha_{n-1})$ as $(y_1, \ldots, y_{2n-1})$, we have that

$$|\, \{j \in \{1, \ldots, (2n-1)\} : y_j \leq \tau\} \,| \geq n.$$

   But this implies that $m^{2n-1}(y_1, \ldots y_{2n-1}) \leq \tau$. That is, $f(\hat{\succ}_D, \tilde{\succ}_{D^c}) \leq f(\tilde{\succ}_D, \tilde{\succ}_{D^c})$, which contradicts our initial hypothesis. □

Thus, falling short of Moulin's (1980) results, Proposition 1 shows that efficient and anonymous generalized median voter schemes are group-strategy-proof (and consequently, strategy-proof) over single-crossing preference profiles, provided that the phantom voters are fixed at the extremes of $\mathbf{R}^*_+$, (i.e., at 0 or $+\infty$).

Interestingly, strategy-proofness cannot be guaranteed in the case of other extended median rules, which allow the socially selected alternative to be the top of a fictitious voter. This conclusion applies also, of course, to the case in which the social choice rule violates the Pareto condition. The following example illustrates this point:[10]

Example 1 *Consider two possible preference profiles* $(\tilde{\succ}_1, \ldots, \tilde{\succ}_n)$ *and* $(\tilde{\succ}_{-n}, \hat{\succ}_n)$ *in* $SC(\tilde{X})$*, and the corresponding collective outcomes* $f^e(\tilde{\succ}) = \tau$ *and* $f^e(\tilde{\succ}_{-n}, \hat{\succ}_n) = \hat{\tau}$*, where* $\hat{\tau} < \tau$*. Suppose that individual preferences are such that, for each individual* $i \in I$*,* $\hat{\tau} \, \tilde{\succ}_i \, \tau$*. For instance, set* $\tilde{\succ}_i = \tilde{\succ}_1$ *for all* $i \in I$*,* $i \neq n$*, with* $\tau(\tilde{\succ}_1) = \hat{\tau}$*, and assume that the true preferences of agent* n *over* $\tilde{X}$*,* $\tilde{\succ}_n$*, are such that* $\tau(\tilde{\succ}_n) > \tau$ *(see Figure 3.1 below). Notice that* τ *does not coincide with the most-preferred alternative of any of the agents. Then, set* $\alpha_1 = \ldots = \alpha_{n-1} = +\infty$*,* $\alpha_n = \tau$ *and* $\alpha_{n+1} = 0$*. It is clear that:*

$$f^e(\tilde{\succ}_1, \ldots, \tilde{\succ}_n) = m^{2n+1}(\underbrace{\hat{\tau}, \ldots, \hat{\tau}}_{n-1 \text{ times}}, \tau(\tilde{\succ}_n), \underbrace{+\infty, \ldots, +\infty}_{n-1 \text{ times}}, \tau, 0) = \tau.$$

Furthermore, it is also evident that the whole coalition I *can improve by declaring* $(\tilde{\succ}_{-n}, \hat{\succ}_n) \in SC(\tilde{X})$*, with* $\tau(\hat{\succ}_n) = \hat{\tau}$*, since*

$$m^{2n+1}(\underbrace{\hat{\tau}, \ldots, \hat{\tau}}_{n \text{ times}}, \underbrace{+\infty, \ldots, +\infty}_{n-1 \text{ times}}, \tau, 0) = \hat{\tau},$$

which is preferred by every coalition member to τ*. That such declaration exists is easy to check. Just consider the case in which agent* n *mimics any of the other agents, so that* $\hat{\succ}_n = \tilde{\succ}_1$*. However, the joint declaration* $(\tilde{\succ}_{-n}, \hat{\succ}_n)$ *implies agent* n *is not revealing honestly his preferences.*

Figure 3.1: Example 1

The reason why strategy-proofness is not preserved in general for the extended median rule, for any possible distribution of the phantoms, is simple. For such arbitrary distributions, the socially selected outcome is not guaranteed to be the most-preferred alternative of a real type. But, without this condition, single-crossing is unable to rule out individual or group manipulations. This is an important difference with single-peakedness, where

[10] Of course, this does not occur if phantoms are not restricted to be at zero or plus infinity, but some or all of them are also allowed to be at the tops of some real voters. However, we ruled out this possibility by considering the phantoms fixed parameters, that do not depend on the preference profile.

strategy-proofness is valid both at the individual and the group level, without any restriction on the values of the phantom voters.

Furthermore, it implies that the family of strategy-proof social choice functions on the domain of single-crossing preference profiles is strictly smaller than the same class on single-peakedness. The rest of the paper is dedicated to prove this result.

Theorem 1 *If $f : SC(\tilde{X}) \to \tilde{X}$ is a tops-only, efficient and strategy-proof social choice function, there exist $\alpha_1, \ldots, \alpha_{n-1} \in \{0, +\infty\}$ such that for every profile $\tilde{\succ} \in SC(\tilde{X})$:*

$$f(\tilde{\succ}) = m^{2n-1}(\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n),\, \alpha_1, \ldots, \alpha_{n-1}). \tag{$*$}$$

Proof Suppose by contradiction that for every combination $\alpha_1, \ldots, \alpha_{n-1} \in \{0, +\infty\}$ there exists a profile $\tilde{\succ} \in SC(\tilde{X})$ such that $f(\tilde{\succ}) \neq m^{2n-1}(\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n), \alpha_1, \ldots, \alpha_{n-1})$. That is equivalent to claim that, if we denote by i^* the i-th position in the order of declarations, for every $i^* = 1, \ldots, n$, there exists a profile $\tilde{\succ} \in SC(\tilde{X})$ such that

$$f(\tilde{\succ}) \;\neq\; \tau(\tilde{\succ}_{i^*}). \tag{$**$}$$

where, as said, agent i^* is the individual whose peak takes up the i-th place (according to the linear order $\leq$) in the distribution of tops $\tau(\tilde{\succ}_{1^*}), \ldots, \tau(\tilde{\succ}_{i-1^*}),\, \tau(\tilde{\succ}_{i^*}),\, \tau(\tilde{\succ}_{i+1^*}), \ldots, \tau(\tilde{\succ}_{n^*})$ generated by the profile $\tilde{\succ} \in SC(\tilde{X})$.[11] Otherwise, if there were a position, say the i-th, such that for every $\tilde{\succ} \in SC(\tilde{X})$, $f(\tilde{\succ}) = \tau(\tilde{\succ}_{i^*})$ we would get a contradiction, since

$$\tau(\tilde{\succ}_{i^*}) = m^{2n-1}(\tau(\tilde{\succ}_{1^*}), \ldots, \tau(\tilde{\succ}_{i^*}), \ldots, \tau(\tilde{\succ}_{n^*}),\, \underbrace{0, \ldots, 0}_{n-j \text{ times}},\, \underbrace{+\infty, \ldots, +\infty}_{j-1 \text{ times}}).$$

Thus, consider the i-th position and a profile $\tilde{\succ} \in SC(\tilde{X})$ verifying $(**)$. Then, if $f(\tilde{\succ}) = x$, $x \neq \tau(\tilde{\succ}_{i^*})$, where $\tau(\tilde{\succ}_{i^*})$ is as before the peak ranked in the i-th place. For an agent k, consider two alternative preferences, $\hat{\succ}_k$ and $\bar{\succ}_k$, such that they verify *simultaneously* the following properties:

Property 1: Both $(\hat{\succ}_k, \tilde{\succ}_{-k})$ and $(\bar{\succ}_k, \tilde{\succ}_{-k})$ are in $SC(\tilde{X})$.
Property 2: $f(\hat{\succ}_k,\, \tilde{\succ}_{-k}) = y \neq x$.

[11] Notice that the i-th position in the above distribution may not be occupied by the agent indexed by i. Single-crossing admits situations where this is the case. Hence, it is important to distinguish between the index of the agent and the position its peak has in the distribution of tops. For notational simplicity we will omit the distinction wherever it is not relevant.

Property 3: $\tau(\bar{\succ}_k) = \tau(\tilde{\succ}_k)$.
Property 4: $f(\hat{\succ}_k, \tilde{\succ}_{-k}) \; \bar{\succ}_k \; f(\tilde{\succ})$.

The existence of a k and the corresponding binary orderings $\hat{\succ}_k$ and $\bar{\succ}_k$ is ensured by Lemmas 1-2 in the Appendix. Therefore we have a pair of preferences $\hat{\succ}_k$ and $\bar{\succ}_k$ such that $(\hat{\succ}_k, \tilde{\succ}_{-k})$, $(\bar{\succ}_k, \tilde{\succ}_{-k}) \in SC(\tilde{X})$, while $f(\hat{\succ}_k, \tilde{\succ}_{-k}) \neq f(\tilde{\succ})$ and $\tau(\bar{\succ}_k) = \tau(\tilde{\succ}_k)$. Since f is tops-only, $f(\tilde{\succ}) = f(\bar{\succ}_k, \tilde{\succ}_{-k})$. Then, $f(\hat{\succ}_k, \tilde{\succ}_{-k}) \; \bar{\succ}_k \; f(\tilde{\succ})$ implies that $f(\hat{\succ}_k, \tilde{\succ}_{-k}) \; \bar{\succ}_k \; f(\bar{\succ}_k, \tilde{\succ}_{-k})$. But this contradicts the assumption that f is strategy-proof. Summarizing, we derived a contradiction from assuming $(**)$. This means there exists a combination $\alpha_1, \ldots, \alpha_{n-1} \in \{0, +\infty\}$ such that $(*)$ holds. □

Let A_f be the range of the social choice function $f : SC(\tilde{X}) \to \tilde{X}$. Then,

Theorem 2 *A social choice function $f : SC(\tilde{X}) \to \tilde{X}$, with range $|A_f| > 2$, is tops-only and strategy-proof if and only if there exists $\alpha_1, \ldots, \alpha_{n-1} \in \{0, +\infty\}$ such that, for all $\tilde{\succ} \in SC(\tilde{X})$,*[12]

$$f(\tilde{\succ}) = m^{2n-1}(\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n), \alpha_1, \ldots, \alpha_{n-1}).$$

Proof
$(\Leftarrow)$: Immediate from Proposition 1.
$(\Rightarrow)$: As the proof of Theorem 1, but using Lemmas 1 and 3 of the Appendix. □

Given a profile $(\succ_1, \ldots, \succ_n) \in SC(X)$ and an arbitrary subset $\tilde{X} \in A(X)$, denote $T(\tilde{X}, \tilde{\succ}) = \{x \in \tilde{X} : \exists\, i \in I$ such that $\tau(\tilde{\succ}_i) = x\}$ the set of all individual peaks in $\tilde{X}$ generated by the induced profile $(\tilde{\succ}_1, \ldots, \tilde{\succ}_n)$:

Corollary 1 *If $f : SC(\tilde{X}) \to \tilde{X}$ is a tops-only and strategy-proof social choice function and $|A_f| > 2$, then $f(\tilde{\succ}) \in T(\tilde{X}, \tilde{\succ})$.*

Proof Immediate from Theorem 2. □

Corollary 2 *If $f : SC(\tilde{X}) \to \tilde{X}$ is a tops-only and strategy-proof social choice function and $|A_f| > 2$, then f is efficient.*

[12] The rules f such that $|A_f| = 1$ are trivially tops-only and strategy-proof, but their (unique) outcomes coincide with those of the extended median rule f^e, with its $n + 1$ phantoms ranging freely over $\mathbf{R}^*_+$. That is, their outcomes may not fall in the restricted class of the tops of the individual preferences.

Proof Trivial. By contradiction, suppose $f : SC(\tilde{X}) \to \tilde{X}$ is a tops-only and strategy-proof social choice function, but assume it is not efficient. Then, there exists a profile $\tilde{\succ} \in SC(\tilde{X})$ and a pair of alternatives $x, y \in \tilde{X}$, $x \neq y$, such that $f(\tilde{\succ}) = x$ while $y \tilde{\succ}_i x$ for every $i \in I$. Therefore, $f(\tilde{\succ}) \notin T(\tilde{X}, \tilde{\succ})$. But this contradicts Corollary 1. □

Proposition 2 *If $f : SC(\tilde{X}) \to \tilde{X}$ is a strategy-proof social choice function, then f is unanimous on its range. That is, for every $\tilde{\succ} \in SC(\tilde{X})$ and any $x \in A_f$, if $\tau(\tilde{\succ}_i) = x \; \forall i \in I$, then $f(\tilde{\succ}) = x$.*

Proof Immediate from the argument $\mathbf{P_2'}$ in Lemma 3 in the Appendix. □

Theorem 3 *A social choice function $f : SC(\tilde{X}) \to \tilde{X}$, with range $|A_f| > 2$, is strategy-proof if and only if there exist $\alpha_1, \ldots, \alpha_{n-1} \in \{0, +\infty\}$ such that, for all $\tilde{\succ} \in SC(\tilde{X})$,*

$$f(\tilde{\succ}) = m^{2n-1}(\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n), \alpha_1, \ldots, \alpha_{n-1}).$$

Proof
($\Leftarrow$): Immediate from Proposition 1.
($\Rightarrow$): As the proof of Theorem 1, but using Lemmas 1 and 4 in the Appendix.
□

Corollary 3 *If $f : SC(\tilde{X}) \to \tilde{X}$ is a strategy-proof social choice function and $|A_f| > 2$, then f is tops-only.*

Proof Trivial. By contradiction, suppose there exists $\hat{\succ}$ and $\tilde{\succ}$ in $SC(\tilde{X})$, such that $\tau(\hat{\succ}_i) = \tau(\tilde{\succ}_i)$ for all $i \in I$, while $f(\hat{\succ}) \neq f(\tilde{\succ})$. By Theorem 3, there exists $\alpha_1, \ldots, \alpha_{n-1} \in \{0, +\infty\}$ such that,

$$f(\tilde{\succ}) = m^{2n-1}(\tau(\tilde{\succ}_1), \ldots, \tau(\tilde{\succ}_n), \alpha_1, \ldots, \alpha_{n-1}).$$

while

$$f(\hat{\succ}) = m^{2n-1}(\tau(\hat{\succ}_1), \ldots, \tau(\hat{\succ}_n), \alpha_1, \ldots, \alpha_{n-1}).$$

Since $\tau(\hat{\succ}_i) = \tau(\tilde{\succ}_i)$ for each $i \in I$, we have that $f(\hat{\succ}) = f(\tilde{\succ})$. Contradiction. □

3.4 Appendix

Lemma 1 *Given* $(\tilde{\Phi}(\theta_1), \ldots, \tilde{\Phi}(\theta_n)) \in SC(\tilde{X})$, *there exists* $k \in I$ *and* $\hat{\theta}_k$, $\bar{\theta}_k \in \Theta \backslash \{\theta_k\}$, $\hat{\theta}_k \neq \bar{\theta}_k$, *such that* $(\tilde{\Phi}(\hat{\theta}_k), \{\tilde{\Phi}(\theta_i)\}_{i \neq k}) \in SC(\tilde{X})$ *and* $(\tilde{\Phi}(\bar{\theta}_k), \{\tilde{\Phi}(\theta_i)\}_{i \neq k}) \in SC(\tilde{X})$, *for any* $\tilde{X} \in A(X)$.

Proof Consider the set $\tilde{X} = \{x_1, \ldots, x_s\}$, $s > 2$, and the profile $(\tilde{\Phi}(\theta_1), \ldots, \tilde{\Phi}(\theta_n)) = (\tilde{\succ}_1, \ldots, \tilde{\succ}_n) \in SC(\tilde{X})$. By contradiction, suppose that $\forall k \in I$, $\hat{\theta}_k, \bar{\theta}_k \in \Theta \backslash \{\theta_k\}$, $\hat{\theta}_k \neq \bar{\theta}_k$, either

$$(\tilde{\Phi}(\hat{\theta}_k), \{\tilde{\Phi}(\theta_i)\}_{i \neq k}) \notin SC(\tilde{X}) \text{ or } (\tilde{\Phi}(\bar{\theta}_k), \{\tilde{\Phi}(\theta_i)\}_{i \neq k}) \notin SC(\tilde{X}). \quad (\star)$$

Let $\Theta^I(\tilde{\succ}) = \{\theta \in \Theta : \exists i \in I \text{ such that } \tilde{\succ}_i = \tilde{\Phi}(\theta)\}$ be the set of *actual* types. For a type $\theta_k \in \Theta^I$, let $L(\theta_k) = \{\theta_i \in \Theta^I : \theta_i < \theta_k\}$ and $H(\theta_k) = \{\theta_i \in \Theta^I : \theta_i > \theta_k\}$. It is straightforward to see that, if $|\Theta^I(\tilde{\succ})| > 2$ it is always possible to find a $\theta_k \in \Theta^I(\tilde{\succ})$ such that $H(\theta_k) \neq \emptyset$ and $L(\theta_k) \neq \emptyset$. Then, define $\theta^{\max} = \min_{(\theta)} H(\theta_k)$ and $\theta^{\min} = \max_{(\theta)} L(\theta_k)$. Clearly, $\tilde{\Phi}(\theta_k)$ and $\tilde{\Phi}(\theta^{\min})$ must differ, as well as $\tilde{\Phi}(\theta_k)$ and $\tilde{\Phi}(\theta^{\max})$. Moreover, $(\tilde{\Phi}(\theta^{\max}), \{\tilde{\Phi}(\theta_i)\}_{i \neq k})$ and $(\tilde{\Phi}(\theta^{\min}), \{\tilde{\Phi}(\theta_i)\}_{i \neq k})$ are in $SC(\tilde{X})$. Therefore, if we define $\bar{\theta}_k$ as $\theta^{\max}$ and $\hat{\theta}_k$ as $\theta^{\min}$, we have a contradiction with $(\star)$.

On the other hand, if $|\Theta^I(\tilde{\succ})| = 1$, it would be trivial to find an individual and a pair of alternative types for this agent, such that the new profiles are still in $SC(\tilde{X})$. Let us, therefore, consider the possibility that $|\Theta^I(\tilde{\succ})| = 2$, i.e. that $\Theta^I(\tilde{\succ}) = \{\theta^1, \theta^2\}$. It is obvious that $\tilde{\Phi}(\theta^1)$ and $\tilde{\Phi}(\theta^2)$ differ in at least a pair of alternatives, say $w > z$. Then, we define $\bar{\theta}$ such that $\tilde{\Phi}(\bar{\theta})$ coincides with $\tilde{\Phi}(\theta^1)$ for every pair of alternatives, except for z and w, and set $w\,\tilde{\Phi}(\bar{\theta})\,z$ if and only if $w\,\tilde{\Phi}(\theta^2)\,z$. If $\tilde{\Phi}(\bar{\theta}) \neq \tilde{\Phi}(\theta^2)$, $\bar{\theta}$ and $\hat{\theta} = \theta^2$ constitutes a pair of alternative types for an agent of type θ^1 that violates $(\star)$. Otherwise, if $\tilde{\Phi}(\bar{\theta}) = \tilde{\Phi}(\theta^2)$, just consider any pair of elements $x, y \in \tilde{X}$, $x > y$ with $x \neq w$ or $y \neq z$ (which exists since $|\tilde{X}| > 2$) for which $x\,\tilde{\Phi}(\theta^2)\,y$. Define $\bar{\theta}'$ such that $y\,\tilde{\Phi}(\bar{\theta}')\,x$. Then $\bar{\theta}'$ and $\hat{\theta} = \theta^1$ constitutes a pair of alternative types for an agent of type θ^2 that, again, violates $(\star)$. If such a pair $\{x, y\}$ does not exists, then $\tilde{\Phi}(\theta^1)$ must be such that $x_1\,\tilde{\Phi}(\theta^1)\,x_2\,\tilde{\Phi}(\theta^1)\,x_3 \ldots x_{s-1}\,\tilde{\Phi}(\theta^1)\,x_s$.[13] But then there must exist a pair $x, y \in \tilde{X}$, say $x > y$, and a type $\bar{\theta}'' \in \Theta$ such that $\tilde{\Phi}(\bar{\theta}'')$ coincides with $\tilde{\Phi}(\theta^2)$, but $y\,\tilde{\Phi}(\theta^2)\,x$ while $x\,\tilde{\Phi}(\bar{\theta}'')\,y$ and $(\tilde{\Phi}(\bar{\theta}''_k), \{\tilde{\Phi}(\theta_i\}_{i \neq k})) \in SC(\tilde{X})$, where k is assumed to be an agent of type θ^2. Again, agent k and the pair $\bar{\theta}''$ and $\hat{\theta} = \theta^1$ contradicts $(\star)$. □

[13] Remember that the set $\tilde{X}$ is such that $x_j \leq x_k$ for $j \leq k$

Lemma 2 *For any efficient and tops-only social choice rule $f : SC(\tilde{X}) \rightarrow \tilde{X}$ that satisfies $(**)$, $\exists\, k \in I$ and $\hat{\succ}_k$ and $\bar{\succ}_k$, such that they verify* simultaneously *Properties 1-4 in the text.*

Proof Suppose, to the contrary, that for every k and every pair of individual preferences $\hat{\succ}_k$, $\bar{\succ}_k$ either:

$\mathbf{P_1}$: At least one of $(\hat{\succ}_k, \tilde{\succ}_{-k})$ or $(\bar{\succ}_k, \tilde{\succ}_{-k})$ is not in $SC(\tilde{X})$; or,
$\mathbf{P_2}$: $f(\hat{\succ}_k, \tilde{\succ}_{-k}) = x$; or,
$\mathbf{P_3}$: $\tau(\bar{\succ}_k) \neq \tau(\tilde{\succ}_k)$; or,
$\mathbf{P_4}$: Either $f(\tilde{\succ}) = f(\hat{\succ}_k, \tilde{\succ}_{-k})$ or $f(\tilde{\succ}) \,\bar{\succ}_k\, f(\hat{\succ}_k, \tilde{\succ}_{-k})$.

Let us consider each possibility in order to get to a contradiction.

$\mathbf{P_1}$: This leads to a contradiction with Lemma 1.

$\mathbf{P_2}$: Suppose that for every k and every $\hat{\succ}_k$, such that $(\hat{\succ}_k,\, \tilde{\succ}_{-k}) \in SC(\tilde{X})$, $f(\hat{\succ}_k, \tilde{\succ}_{-k}) = x$. In words, this means that no individual deviation from $\tilde{\succ} = (\tilde{\succ}_1, \ldots, \tilde{\succ}_n)$ matters. This implies by induction that $f(\hat{\succ}_{I'}, \tilde{\succ}_{I''}) = f(\tilde{\succ})$, where $(I^{'}, I^{''})$ is any partition of the set of agents. In effect, the base case, where $I^{'} = \{k\}$ and $I^{''} = I \setminus \{k\}$ is in fact our hypothesis: $f(\hat{\succ}_k, \tilde{\succ}_{-k}) = f(\tilde{\succ})$. Now, suppose that $f(\hat{\succ}_{I'}, \tilde{\succ}_{I''}) = f(\tilde{\succ})$ where $k \in I^{''}$. Then, by transitivity, $f(\hat{\succ}_{I'}, \tilde{\succ}_{I''}) = f(\hat{\succ}_k, \tilde{\succ}_{-k})$. We will show that $f(\hat{\succ}_{I'+k}, \tilde{\succ}_{I''-k}) = f(\tilde{\succ})$. Suppose, to the contrary, that after a new deviation, $f(\hat{\succ}_{I'+k}, \tilde{\succ}_{I''-k}) \neq f(\tilde{\succ})$, i.e. that $f(\hat{\succ}_{I'+k}, \tilde{\succ}_{I''-k}) \neq f(\hat{\succ}_{I'}, \tilde{\succ}_{I''})$. Without loss of generality, assume that $f(\hat{\succ}_{I'}, \tilde{\succ}_{I''}) < f(\hat{\succ}_{I'+k}, \tilde{\succ}_{I''-k})$. Suppose k and $\hat{\succ}_k$ are such that either $f(\hat{\succ}_{I'+k}, \tilde{\succ}_{I''-k}) \;\tilde{\succ}_k\; f(\hat{\succ}_{I'}, \tilde{\succ}_{I''})$ or $f(\hat{\succ}_{I'}, \tilde{\succ}_{I''}) \;\hat{\succ}_k\; f(\hat{\succ}_{I'+k}, \tilde{\succ}_{I''-k})$. Notice that a k verifying this may exist, since $\hat{\succ}_k$ and $\tilde{\succ}_k$ should differ in the valuation of at least a pair of points. Then, it is immediate to see that either case implies a violation of strategy-proofness. On the contrary, assume that for every k and every $\hat{\succ}_k$ both $f(\hat{\succ}_{I'}, \tilde{\succ}_{I''}) \;\tilde{\succ}_k\; f(\hat{\succ}_{I'+k}, \tilde{\succ}_{I''-k})$ and $f(\hat{\succ}_{I'+k}, \tilde{\succ}_{I''-k}) \;\hat{\succ}_k\; f(\hat{\succ}_{I'}, \tilde{\succ}_{I''})$. It is easy to find a k and an ordering $\hat{\succ}_k$ such that these conditions are not verified simultaneously. Therefore, we have proved our inductive hypothesis, i.e. that $f(\hat{\succ}_{I'+k}, \tilde{\succ}_{I''-k}) = f(\tilde{\succ})$ for every partition $(I^{'}, I^{''})$ of the set of agents and every $k \in I^{''}$. In the limit, we have that $f(\hat{\succ}) = f(\tilde{\succ})$. Once achieved this limit, consider the case in which preferences are identical for all agents. Concretely, take the profile $\tilde{\succ}$. Define the permutation $\sigma : I \rightarrow I$ such that, for every $i,\, l \in I$, $\sigma(i) = \sigma_i < \sigma_l = \sigma(l)$ if $\theta_i < \theta_l$; and, if $\theta_i = \theta_l$ and $l < i$, set $\sigma(l) > \sigma(i)$. To avoid to work explicitly with the permutation, in what follows there is no confusion in supposing that the index of each individual refers to his new number under the permutation. Now choose sequentially $\hat{\succ}_k = \tilde{\succ}_{i^*}$ for each agent $k = i^*+1, i^*+2, \ldots, n^*$ and then for $k = i^*-1, i^*-2, \ldots, 1^*$. (Remember that we were considering the i-th positional dictator choice rule). By $(**)$, $f(\tilde{\succ}) \neq \tau(\tilde{\succ}_{i^*})$. Therefore, $\tau(\tilde{\succ}_{i^*})\,\hat{\succ}_k\, f(\hat{\succ})$ for every $k \in I$, contradicting the fact that f is Pareto efficient.

$\mathbf{P_3}$: Suppose, by contradiction, that for every k and every $\bar{\succ}_k$, such that $(\bar{\succ}_k, \tilde{\succ}_{-k}) \in SC(\tilde{X})$, $\tau(\bar{\succ}_k) \neq \tau(\tilde{\succ}_k)$. It is immediate to see that such statement is false. Just consider the profile $(\tilde{\succ}_1, \ldots, \tilde{\succ}_n)$ and the type $\theta^{min} = \min_{(\theta)}\ \Theta^I(\tilde{\succ})$. Then, for an individual i of type θ^{min}, it is always possible to define a preference relation $\bar{\succ}_i$ such that $(\bar{\succ}_i, \tilde{\succ}_{-i}) \in SC(\tilde{X})$, $\tau(\bar{\succ}_i) = \tau(\tilde{\succ}_i)$, and $\bar{\succ}_i$ and $\tilde{\succ}_i$ differ in the ranking of at least a pair of distinct alternatives $x, y \in \tilde{X}\backslash\{\tau(\bar{\succ}_i), \tau(\tilde{\succ}_i)\}$.

$\mathbf{P_4}$: Suppose, by contradiction, that for every k and every two preference orderings $\hat{\succ}_k$ and $\bar{\succ}_k$, such that $(\hat{\succ}_k, \tilde{\succ}_{-k}), (\bar{\succ}_k, \tilde{\succ}_{-k}) \in SC(\tilde{X})$, either $f(\tilde{\succ}) = f(\hat{\succ}_k, \tilde{\succ}_{-k})$ or $f(\tilde{\succ})\,\bar{\succ}_k\, f(\hat{\succ}_k, \tilde{\succ}_{-k})$. Since we have already contradicted $\mathbf{P_2}$, the first possibility is ruled out. Thus, without loss of generality, assume $f(\tilde{\succ}) < f(\hat{\succ}_k, \tilde{\succ}_{-k})$. By single-crossing, $f(\tilde{\succ})\,\bar{\succ}_k\, f(\hat{\succ}_k, \tilde{\succ}_{-k}) \Rightarrow f(\tilde{\succ})\,\tilde{\succ}_i\, f(\hat{\succ}_k, \tilde{\succ}_{-k})$ for every i such that $\theta_i \leq \bar{\theta}_k$, where θ_i and $\bar{\theta}_k$ are such that $\tilde{\Phi}(\theta_i) = \tilde{\succ}_i$ and $\tilde{\Phi}(\bar{\theta}_k) = \bar{\succ}_k$, respectively. On the other hand, $f(\hat{\succ}_k, \tilde{\succ}_{-k})\,\hat{\succ}_k\, f(\tilde{\succ})$. Otherwise, k can manipulate f at $(\hat{\succ}_k, \tilde{\succ}_{-k})$ via $\tilde{\succ}_k$. Combining this with the previous claim, it follows that $\hat{\theta}_k > \bar{\theta}_k$. But then there must exist a type $\bar{\theta}_k^{'} \in \Theta$, $\bar{\theta}_k < \bar{\theta}_k^{'} < \hat{\theta}_k$, such that the associated ordering $\tilde{\Phi}(\bar{\theta}_k^{'}) = \bar{\succ}_k^{'}$ coincides with $\bar{\succ}_k$ except in the ranking of the alternatives $f(\tilde{\succ})$ and $f(\hat{\succ}_k, \tilde{\succ}_{-k})$. That is, $f(\hat{\succ}_k, \tilde{\succ}_{-k})\,\bar{\succ}_k^{'}\, f(\tilde{\succ})$. Thus, the pair $\hat{\succ}_k$ and $\bar{\succ}_k^{'}$ contradicts $\mathbf{P_4}$.

Thus, since we disproved $\mathbf{P_1} - \mathbf{P_4}$, there must exist a k with two alternative preferences, $\hat{\succ}_k$ and $\bar{\succ}_k$ that verifies simultaneously Properties $1-4$. □

Lemma 3 *For any tops-only social choice rule $f : SC(\tilde{X}) \rightarrow \tilde{X}$ that satisfies $(**)$ and $A_f > 2$, $\exists\, k \in I$ and $\hat{\succ}_k$ and $\bar{\succ}_k$, such that they verify* simultaneously *Properties 1-4 in the text.*

Proof Suppose, as in the proof of Lemma 2, that for every k and every pair of individual preferences $\hat{\succ}_k$, $\bar{\succ}_k$ either:

$\mathbf{P_1}$: At least one of $(\hat{\succ}_k, \tilde{\succ}_{-k})$ or $(\bar{\succ}_k, \tilde{\succ}_{-k})$ is not in $SC(\tilde{X})$; or,

$\mathbf{P_2'}$: $f(\hat{\succsim}_k, \tilde{\succsim}_{-k}) = x$; or,
$\mathbf{P_3}$: $\tau(\bar{\succsim}_k) \neq \tau(\tilde{\succsim}_k)$; or,
$\mathbf{P_4}$: Either $f(\tilde{\succsim}) = f(\hat{\succsim}_k, \tilde{\succsim}_{-k})$ or $f(\tilde{\succsim}) \,\bar{\succ}_k\, f(\hat{\succsim}_k, \tilde{\succsim}_{-k})$.

The arguments in the proof of Lemma 2 can be repeated to show that $\mathbf{P_1}$, $\mathbf{P_3}$ and $\mathbf{P_4}$ are false. To show that $\mathbf{P_2'}$ is also false consider the following argument:

$\mathbf{P_2'}$: For expositional simplicity, consider the *leftist choice rule*, instead of the i-th positional dictator choice rule. That is, take the combination of αs in $\{0, +\infty\}^{n-1}$ that always chooses the smallest reported peak of a real voter. Suppose the profile $(\tilde{\succsim}_1, \ldots, \tilde{\succsim}_n) \in SC(\tilde{X})$ is such that $f(\tilde{\succsim}) = x$, while $x \neq \tau(\tilde{\succsim}_i)$, where i is the agent which has the first ranked peak in the distribution $\tau(\tilde{\succsim}_1), \ldots, \tau(\tilde{\succsim}_n)$. After renaming the agents as in $\mathbf{P_2}$, assume that, for every k and every $\hat{\succsim}_k$, such that $(\hat{\succsim}_k, \tilde{\succsim}_{-k}) \in SC(\tilde{X})$, $f(\hat{\succsim}_k, \tilde{\succsim}_{-k}) = f(\tilde{\succsim}) = x$. By the same reasoning applied in $\mathbf{P_2}$, it follows that $f(\hat{\succsim}_{I_\sigma'}, \tilde{\succsim}_{I_\sigma''}) = f(\tilde{\succsim})$ for every partition (I_σ', I_σ'') of the set of agents. On the other hand, since $|A_f| > 2$, there exist $y \in \tilde{X}$, $y \neq x$, and $(\tilde{\succsim}_1', \ldots, \tilde{\succsim}_n') \in SC(\tilde{X})$ such that $f(\tilde{\succsim}_1', \ldots, \tilde{\succsim}_n') = y$. We want to prove that, after a finite number of individual deviations from $(\tilde{\succsim}_1, \ldots, \tilde{\succsim}_n)$, we can achieve a profile $(\hat{\succsim}_1, \ldots, \hat{\succsim}_n) \in SC(\tilde{X})$ such that $\tau(\hat{\succsim}_i) = \tau(\tilde{\succsim}_i') \; \forall i \in I_\sigma$, while $f(\hat{\succsim}) \neq f(\tilde{\succsim}')$. To do that, consider the most leftist type $\theta^1 \in \Theta$, characterized by the binary relation $x_1 \,\tilde{\Phi}(\theta^1)\, x_2 \ldots x_{s-1} \,\tilde{\Phi}(\theta^1)\, x_s$. Then, by sequentially deviating each agent $k = 1, \ldots, n$ from $\tilde{\succsim}_k$ to $\tilde{\Phi}(\theta^1)$, we obtain the unanimous profile $(\tilde{\Phi}(\theta^1), \ldots, \tilde{\Phi}(\theta^1))$, which is obviously in $SC(\tilde{X})$. Moreover, $f(\tilde{\Phi}(\theta^1), \ldots, \tilde{\Phi}(\theta^1)) = x$. But now we can take individual in the n-th position and define a sequence of deviations for this agent $\hat{\succsim}_n^1, \ldots, \hat{\succsim}_n^h$, where $\hat{\succsim}_n^1$ is obtained from $\tilde{\Phi}(\theta^1)$ by moving up to the first position (to the top) the greatest alternative in $T(\tilde{X}, \tilde{\succsim}')$; $\hat{\succsim}_n^2$ is obtained from $\hat{\succsim}_n^1$ by moving up to the second position the second higher alternative in $T(\tilde{X}, \tilde{\succsim}')$; etc. Clearly, the profile $(\tilde{\Phi}(\theta^1), \ldots, \tilde{\Phi}(\theta^1), \tilde{\Phi}(\hat{\theta}_n^j)) \in SC(\tilde{X})$, for each $j = 1, \ldots, h$, where $\tilde{\Phi}(\hat{\theta}_n^j) = \hat{\succsim}_n^j$. Moreover, $f(\tilde{\Phi}(\theta^1), \ldots, \tilde{\Phi}(\theta^1), \tilde{\Phi}(\hat{\theta}_n^j)) = x$. Denote $\hat{\succsim}_n^h = \hat{\succsim}_n$. Consider now the individual in the $(n-1)$-th position. Define $\hat{\succsim}_{n-1}^1, \ldots, \hat{\succsim}_{n-1}^{h-1}$, where $\hat{\succsim}_{n-1}^1$ is obtained from $\tilde{\Phi}(\theta^1)$ by moving up to the first position (to the top) the second greatest alternative in $T(\tilde{X}, \tilde{\succsim}')$; $\hat{\succsim}_{n-1}^2$ is obtained from $\hat{\succsim}_{n-1}^1$ by moving up to the second position the third higher alternative in $T(\tilde{X}, \tilde{\succsim}')$; etc. After repeating this process for each agent, we finally reach individual in the 1-st for which we simply define an alternative ordering $\hat{\succsim}_1$ that moves up to the top the smallest element in $T(\tilde{X}, \tilde{\succsim}')$. Thus, by proceeding in this way, we derive a profile $(\tilde{\Phi}(\hat{\theta}_1), \ldots, \tilde{\Phi}(\hat{\theta}_n)) \in SC(\tilde{X})$ such that $\tau(\tilde{\Phi}(\hat{\theta}_i)) = \tau(\tilde{\succ}_i')$, but $f(\hat{\succ}_1, \ldots, \hat{\succ}_n) = x \neq f(\tilde{\succ}_1', \ldots, \tilde{\succ}_n')$, contradicting the tops-only condition. Thus $\mathbf{P_2'}$ is also false.[14] $\square$

Lemma 4 *For any social choice rule $f : SC(\tilde{X}) \to \tilde{X}$ that satisfies* $(**)$ *and $|A_f| > 2$, $\exists\, k \in I$ and $\hat{\succ}_k$ and $\bar{\succ}_k$, such that they verify* simultaneously *Properties 1-4 in the text.*

Proof Suppose, as in the proof of Lemma 2 and 3, that for every k and every pair of individual preferences $\hat{\succ}_k$, $\bar{\succ}_k$ either:

$\mathbf{P_1}$: At least one of $(\hat{\succ}_k, \tilde{\succ}_{-k})$ or $(\bar{\succ}_k, \tilde{\succ}_{-k})$ is not in $SC(\tilde{X})$; or,
$\mathbf{P_2''}$: $f(\hat{\succ}_k, \tilde{\succ}_{-k}) = x$; or,
$\mathbf{P_3}$: $\tau(\bar{\succ}_k) \neq \tau(\tilde{\succ}_k)$; or,
$\mathbf{P_4}$: Either $f(\tilde{\succ}) = f(\hat{\succ}_k, \tilde{\succ}_{-k})$ or $f(\tilde{\succ}) \bar{\succ}_k f(\hat{\succ}_k, \tilde{\succ}_{-k})$.

Following the same arguments that before, it is immediate to prove that $\mathbf{P_1}$, $\mathbf{P_3}$ and $\mathbf{P_4}$ are false. To show that $\mathbf{P_2''}$ is also false consider again the argument for $\mathbf{P_2}$ in the proof of Lemma 2. Recall that we have a profile $(\hat{\succ}_{\sigma_1}, \ldots, \hat{\succ}_{\sigma_n})$ that obtains from $\tilde{\succ}$ by means of a sequence of deviations, such that $f(\hat{\succ}_{\sigma_1}, \ldots, \hat{\succ}_{\sigma_n}) = f(\tilde{\succ})$. In particular, this profile can be such that $\hat{\succ}_{\sigma_j} = \hat{\succ}_{\sigma_l}$ for every pair $j, l \in I$. Moreover, it is possible to choose, for each $\sigma_i \in I_\sigma$, $\tau(\hat{\succ}_{\sigma_i}) = y \neq x$, since $|A_f| > 2$. But then we have, on one hand that $f(\hat{\succ}_{\sigma_1}, \ldots, \hat{\succ}_{\sigma_n}) = x$ while on the other,$f(\hat{\succ}_{\sigma_1}, \ldots, \hat{\succ}_{\sigma_n}) = y$ (by Proposition 2). Contradiction. Then, $\mathbf{P_2''}$ is also false. $\square$

[14]Notice that, to get to a contradiction, it is crucial to choose $x \neq \tau(\succ_1)$. Otherwise, agent 1 can manipulate f at $(\hat{\succ}_1, \ldots, \hat{\succ}_n)$ via $\tilde{\Phi}(\theta^1)$, violating the strategy-proofness of f.

Chapter 4

Application: repeated games and the agenda-setter model

4.1 Introduction

A dominant paradigm in the study of political resource allocation is the Median Voter Theorem (henceforth MVT).[1] Its huge acceptance within economic applications is due to the fact that the MVT, as the notion of perfect competition in the study of market resource allocations, greatly simplifies the analysis of one-dimensional collective decision-making problems. In effect, within this model the collective outcomes are determined exclusively by the preferences of the median voter. All other institutional details and diversity of tastes become completely irrelevant for characterizing equilibrium policies.

The underlying logic of this prediction stems from an implicit competitive or decentralized model of proposal-making, in which all individuals are allowed to make proposals and any alternative other than the median ideal point can be defeated under the majority rule by a policy closer to the median. The usefulness of the MVT depends critically on the existence of such competitive agenda-setting process.

Romer and Rosenthal (1978, 1979a) made the first formal investigation of the implications of agenda control on collective decisions, in a model commonly known as the *agenda-setter model*. Rather than focusing on the effect of competition on policy outcomes, Romer and Rosenthal assumed the exis-

[1] There are basically two versions of the MVT. One version comes from Black (1948), for voting in committees, and Downs (1957), for electoral competition, and it depends on the assumption of single-peaked preferences. The other one, coming from Rothstein (1991), Gans and Smart (1996), and Austen-Smith and Banks (1999) assumes order-restricted preferences. For more on this, see chapter 2 in this book.

tence of an *agenda-setter* endowed with the ability to set the voting agenda. Offering to the electorate a choice between an exogenously determined reversion level and an alternative selected by him, they showed that an agenda-setter with strictly increasing preferences over the one-dimensional policy space can deviate the social outcome from the median voter's most-preferred alternative. They also established equilibrium predictions that indicate how this bias could be quite severe depending on the status quo location.

This chapter extends the static agenda-setter model to an infinite-horizon framework. The main purpose is to determine whether the repetition of the policy-making process and the strategic (forward-looking) behavior of the electorate could modify the main prediction stated by Romer and Rosenthal; namely that when the environment allows monopoly agenda-setting power the median voter's ideal policy cannot emerge as the equilibrium outcome of the policy game. Using the theoretical results of the literature on repeated games, this chapter shows that, even when the institutional structure is not of the arrovian type, in the sense that the proposal-making process is not fully decentralized, the incentives produced by the dynamic interaction could be sufficiently strong to induce the kind of cooperation needed to support preference-based equilibria.

The chapter is organized as follows. Section 4.2 presents the one-shot version of the agenda-setter model, where two long-run players, the agenda-setter and a representative voter, interact to determine a policy outcome from a one-dimensional policy space. After establishing a restriction over individual preferences that guarantees the existence of a representative voter, this section proves the existence of a unique subgame perfect equilibrium in pure strategies, with the property that the alternative socially selected exceeds the median voter's most-preferred policy. Section 4.3 extends the basic model to analyze the effect of time on the equilibrium policy. The main finding of this section is that the median voter's ideal point could still be supported as a subgame perfect equilibrium of the repeated agenda control game. A profile of strategies that leads to such outcome and that removes the myopic behavior of the median voter is also established. Finally, conclusions are stated in section 4.4.

4.2 The one-shot model

Consider a collective decision-making organization (for example, a legislature) composed by a finite set of rational and infinite-lived agents $N \cup \{A\}$, $|N| > 2$ odd, where N represents the set of voters and A the agenda-setter. Let $X \subset \Re_+$, $|X| > 2$, denote a compact subset of feasible policies and

$Q = [\underline{q}, \bar{q}] \subseteq X$ the set of admissible values of a random variable called the reversion point or the status quo.

The agenda setter's preferences over the set of alternatives, u_A, are assumed to be strictly increasing on X. That is,

$$\forall x,\, y \in X,\ u_A(x) > u_A(y) \ \Leftrightarrow\ x > y. \tag{4.1}$$

On the other hand, for all $i \in N$, voter i's preferences over X, u_i, are assumed to be single-peaked and symmetric around its most-preferred alternative.[2] Concretely, $u_i(x) = -(x - \theta_i)^2$, which means that,

$$\forall i \in N,\ x,\, y \in X,\ u_i(x) \geq u_i(y) \ \Leftrightarrow\ \|x - \theta_i\| \leq \|y - \theta_i\|, \tag{4.2}$$

where $\|\cdot\|$ denotes the standard Euclidean metric and θ_i voter i's ideal point. Together these assumptions are referred to as Euclidean preferences, meaning that outcomes equidistant from i's ideal point are judged to be indifferent by i.[3]

The model considers a collective choice problem in which the agenda-setter and voters interact to determine a policy outcome in X. The timing of events is as follows. In each period, the agenda-setter observes the realized status quo $q \in Q$, previously selected by Nature, and then he offers a proposal $\pi(q) \in X$ to the electorate. The body observes the status quo and then decides whether to accept or not the agenda-setter's proposal, using for that the majority rule. If $\pi(q)$ is rejected, then the reversion outcome is implemented.

4.2.1 Representative voter

As it is usual in the applications of the agenda-setter model, this chapter retains the analytically convenient feature of reducing the electorate to a single voter. That is, the political interaction described above will be analyzed in terms of a game with only two players: the agenda-setter and a *representative voter*.[4] Other related papers that employ a similar approach are Romer

[2]Single-peakedness constitutes one of the most important classes of domain restrictions where non-dictatorial preference aggregation processes are possible. In the particular case of pairwise majority voting, it guarantees the existence of a well-defined equilibrium that coincides with the most-preferred alternative of the median voter. For a formal definition, see chapter 2.

[3]Euclidean preferences drive many important results in the literature of political economy. See, for example, McKelvey (1976), Laver and Shepsle (1990), Ferejohn and Krehbiel (1987), Koford (1989), Milyo (1999), etc.

[4]The representative voter is an individual whose strict preference for any alternative x over any alternative y implies: (1) x strictly defeats by majority rule, if there are an odd

and Rosenthal (1978, 1979a), Ingberman (1985), Rosenthal (1990), Banks (1990, 1993), etc. However, an important difference with these works is that here we make explicit the kind of restriction that we need to impose over the preferences in order to make this simplification.

Concretely, such procedure is justified, for example, in one-dimensional problems, with single-peaked preferences. In those cases, the Median Voter Theorem predicts that, if the proposal-making mechanism is decentralized, then the median voter's ideal point has a strict simple majority over every other proposal. This means that it is sufficient to identify the median of the most-preferred alternatives to perfectly characterize the core of the majority rule. The collective choice problem is then reduced to a particular individual's choice problem.

However, even in those cases, proving the existence of equilibria and characterizing equilibrium outcomes are only the first step in developing a useful predictive theory of voting. In applications, one would also like to do comparative statics exercises with respect to parameter changes, which requires more information than the set of majority winners. When parameters change, the identity of the pivotal voter may change as well, and then equilibrium outcomes could vary in unpredictable ways. For this reason, comparative statics calculations conducted for single-peaked preferences generally examine parameter changes for which voters unanimously agree on the desired direction of change.

Besides, even if we are not interested in comparative statics exercises, except in the special case where the institutional environment allows proposals by all individuals or a pure Hotelling-Downs competition, the median voter's ideal point may not appear on a ballot. That is, voters could be forced to choose in agendas where none of the alternatives represent the median ideal top or even any other individual peak. In those cases, it follows immediately that, even if we can identify the median voter and model his tastes, we will not be able to use this information to predict majority-voting equilibria.[5] The following example illustrates this problem.

Suppose there exists three individuals with preferences as depicted in Figure 4.1. Let $s_i(q)$ denote the greatest policy voter i will find as good as

number of voters; and (2) x weakly defeats y otherwise. This result holds by the median voter if x is his ideal point or if preferences satisfy a generalized symmetry property, but not in general. We will return to this point later in the text.

[5]For example, in the institutional structure detailed in Romer and Rosenthal (1978, 1979a), agenda-setting power allows politicians to obtain outcomes that differ systematically from the median ideal point.

the reversion point q,

$$s_i(q) = \sup_x \{x \in X : \ u_i(x) \geq u_i(q)\}, \tag{4.3}$$

and define the median voter m as the person who owns the median ideal point θ_m,[6]

$$\theta_m = \theta^m \equiv med\,\{\theta_i\}_{i \in N}. \tag{4.4}$$

Since preferences are single-peaked on $\Re_+$, the MVT predicts that m's ideal peak is supported by a majority coalition. The pivotal role played by m is easily understood:[7] at least half of the electorate agrees with the median that θ_m is preferred to any other alternative, and this is valid only for such alternative and, therefore, for no other individual.

In contrast, suppose the institutional structure allows agenda-setting power, as in Romer and Rosenthal (1978, 1979a). More precisely, assume that voters face a "take-it-or-leave-it" vote between an exogenously determined status quo q and a policy outcome $\pi(q)$, selected by the agenda-setter. Given (4.1), it is easy to prove that the equilibrium policy will be (see Proposition 1 below),

$$\pi^* = med\,\{s_i\}_{i \in N}. \tag{4.5}$$

Then, as long as π^* could be different from s_m, the identity of the pivotal voter will not generally coincide with the median one. In Figure 4.1, $s_m = s_2$, but $\pi^* = s_3$.

Figure 4.1: Median vs. pivotal voter

As the reader may anticipate, the fact that the identity of the pivotal voter could vary with the subset of feasible policies considered (which is in turn determined by the institutional structure) represents an unfortunate result. In few words, the general implication behind the example is that we should not expect the median voter to be decisive in any subset of policy alternatives, including those in which his ideal peak is not available.

[6] For any odd positive integer r, let $x = \{x_1, x_2, \ldots, x_r\}$ be a finite sequence of X. The median of x, denoted by $x^m \equiv med\,\{x_i\}_{i=1}^r$, is such that $|\{x_i : x_i \leq x^m\}| \geq (r+1)/2$ and $|\{x_i : x^m \leq x_i\}| \geq (r+1)/2$.

[7] The pivotal voter must be differentiated from the representative one. The pivotal voter is the person whose vote determines the electoral result. As Ingberman (1985) pointed out, while the representative voter is not necessarily unique, the pivotal does.

For that being the case, Rothstein (1991) has shown that voters' preferences must satisfy additional properties, such that the ones he called *order-restriction*.[8] Unlike single-peakedness, order-restriction imposes limitations on the character of voter heterogeneity, rather than on the shape of individual preferences. Under order-restricted preferences, individuals are assigned a position along a left-right scale with the condition that, for any pair of alternatives, the set of individuals preferring one of the alternatives all lie to one side of those who prefer the other.

More technically, for any two sets of integers A and B, let $A \gg B$, read "A is higher than B", if for every $a \in A$ and $b \in B$, $a > b$.

Definition 1 *A preference profile* $(u_1, \ldots, u_N)$ *is order-restricted on* X *if and only if there exists a permutation* $\gamma : N \rightarrow N$ *such that for all distinct pair of alternatives* x, $y \in X$, *either* [9]

$$\{i \in N : u_i(x) > u_i(y)\} \gg \{i \in N : u_i(x) = u_i(y)\} \gg \{i \in N : u_i(x) < u_i(y)\}$$

or

$$\{i \in N : u_i(x) > u_i(y)\} \ll \{i \in N : u_i(x) = u_i(y)\} \ll \{i \in N : u_i(x) < u_i(y)\}.$$

Let u_S indicate the social preference ordering induced by the majority rule, such that for all x, $y \in X$, $u_S(x) \geq u_S(y)$ if and only if $|\{i \in N : u_i(x) > u_i(y)\}| \geq |\{j \in N : u_j(y) > u_j(x)\}|$. Define the (possibly non unique) representative voter as any individual i for whom, for all x, $y \in X$, $u_i(x) > u_i(y)$ implies $u_S(x) \geq u_S(y)$. Then, Rothstein's (1991) Representative Voter Theorem ensures that, as long as preferences satisfy order-restriction, the median voter m is also a representative voter.[10] This means that, for any pair of alternatives x and y, not just for the median top θ_m, say $x < y$, if the median voter prefers x, then all voters to his left agree with him; and, if the median voter prefers y, then all voters to his right agree also with him.

In the following proposition we show that, under our assumption on voters' preferences, this condition holds.

[8] This constraint holds in papers by Roberts (1977), Grandmont (1978), Beck (1978) and a number of others. Fundamentally, in these models there is a natural ordering of individuals, not the alternatives, and relative to this ordering the preferences of individuals over any pair of alternatives obey a simple non-reswitching rule (see Definition 1 in the text).

[9] Since it is cumbersome to work explicitly with the permutation, in what follows there is no confusion in supposing that the number of each individual refers to his new number under the permutation.

[10] For more details see Rothstein (1991).

Proposition 1 *If individual preferences are single-peaked on X and symmetric around their ideal points, then they satisfy order-restriction on X.*[11]

PROOF Assume by contradiction that $(u_1, u_2, \ldots, u_N)$ does not satisfy order-restriction on X. Define a permutation $\gamma^* : N \to N$ in the following way: for all $i, j \in N$, let $\gamma^*(i) > \gamma^*(j)$ if $\theta_i > \theta_j$; on the other hand, if $\theta_i = \theta_j$ and $i < j$, set $\gamma^*(i) < \gamma^*(j)$. Since N is a finite set of natural numbers and $\theta_i \in \Re_+$ is unique for each $i \in N$, it is easy to verify that γ^* is always well-defined. Without lost of generality, consider an arbitrary pair $x, y \in X$, say $x \leq y$, and define the "cut point" $\theta^* = (y - x)/2$. Let N^+ and N^- be defined as follows:

$$N^+ = \{i \in N : \theta_i > \theta^*\}, \tag{4.6}$$

and

$$N^- = \{j \in N : \theta_j < \theta^*\}. \tag{4.7}$$

Under γ^*, it is clear that $N^- \ll N^+$. Besides, (4.2) implies

$$u_i(y) > u_i(x) \; \forall i \in N^+, \tag{4.8}$$

and

$$u_j(y) < u_j(x) \; \forall j \in N^-. \tag{4.9}$$

On the other hand, if there exists $k \in N$ such that $\theta_k = \theta^*$, then again by (4.2) $u_k(x) = u_k(y)$. Moreover, θ_k must be unique and, therefore, $N^- \ll \{i \in N : u_i(x) = u_i(y)\} \ll N^+$. Hence, from (4.8) and (4.9) it follows that Definition 1 holds, contradicting our hypothesis that $(u_1, \ldots, u_N)$ does not satisfy order-restriction on X. $\square$

Using the primitives of our model, Proposition 1 guarantees that voters' preferences satisfy order-restriction and, therefore, it allows to apply the Representative Voter Theorem. However, this Theorem, as well as the MVT, is a result derived assuming *sincere voting*. That is, it is based on the hypothesis that in every vote each citizen votes for the alternative that gives him the highest utility according to his policy preferences.

Obviously, this may not be always the case. Perhaps it is reasonable to suppose that in cases where voters meet infrequently, so that they do not learned how to act other than sincerely; or where they know little about each other's preferences, and hence cannot fully anticipate the consequences of their actions. But in professional committees, for example, such that those found in legislatures, it is clearly an unreasonable assumption.

[11] A similar result was derived previously by Enelow and Hinich (1984), Rothstein (1991) and Gans and Smart (1996).

Fortunately, the results established in the previous two chapters allow to use the Representative Voter Theorem in strategic settings.

4.2.2 Equilibria

Now we complete the description of the game. In order to do that, define a proposal strategy for A as a function $\pi : Q \to X$ and a response strategy for m as a function $\omega : Q \times X \to \{0, 1\}$, where $\omega(q, \pi) = 1$ indicates that the median voter accepts the alternative chosen by A and $\omega(q, \pi) = 0$ denotes his rejection. Assume also that $\Pi = \{\pi : \pi(q) \in X\}$ and $\Omega = \{\omega : \omega(q, \pi) \in \{0, 1\}\}$ characterize the strategic sets of both players, and $v_i : \Pi \times \Omega \to \Re_+$ the continuous payoff function of agent i, $i = m, A$:

$$v_i(\pi, \omega) = \begin{cases} u_i(\pi) & \text{if } \pi \text{ is accepted,} \\ u_i(q) & \text{otherwise.} \end{cases} \tag{4.10}$$

The timing of the events is as follows: the agenda-setter makes a proposal $\pi(q)$ from X, which the median voter either accepts or rejects. If m accepts the agenda-setter's alternative, it becomes the policy outcome. Otherwise, the result is equal to the reversion level q, where q is a random variable with positive density $g(\cdot)$ on Q.[12] In this version of the agenda-setter model, both players know the true level of q before making their choices.

Let $s_i(q)$ be defined as in (4.3). That is, let $s_i(q)$ indicate the greatest policy voter i will support given the status quo q.

Lemma 1 *$s_i(q)$ is increasing in θ_i. That is, for all $q \in Q$, i, $j \in N$,*

$$\theta_i > \theta_j \;\Rightarrow\; s_i \geq s_j.$$

Proof. Consider any two individuals i, j in N. Suppose, without lost of generality, $\theta_i < \theta_j$. Three cases are possible:

- If $q < \theta_i$, then $\|q - \theta_i\| < \|q - \theta_j\|$. By definition, $\|q - \theta_i\| = \|s_i - \theta_i\|$ and $\|q - \theta_j\| = \|s_j - \theta_j\|$. Thus, $\|s_i - \theta_i\| < \|s_j - \theta_j\|$. But, since single-peakedness implies $s_i > \theta_i$ and $s_j > \theta_j$, it follows that $s_i < s_j$.
- If $\theta_i \leq q < \theta_j$, then $s_i = q$ and, by the same reasoning applied before, $s_j > \theta_j$. Thus, $s_i < s_j$.
- Finally, if $\theta_j \leq q$, then $s_i = s_j = q$. □

[12]Without important consequences, we arbitrarily assume a "yea" vote in the case of indifference; that is, whenever the median is indifferent between the agenda-setter's proposal and the status quo.

Proposition 2 *The game* $G = \langle (A, m); (\pi, \omega); (v_A, v_m) \rangle$ *has a unique subgame perfect equilibrium in pure strategies, where the equilibrium policy* π^* *is given by:*

$$\pi^*(q) = \begin{cases} 2\theta_m - q & \text{if } q < \theta_m, \\ q & \text{otherwise.} \end{cases} \tag{4.11}$$

Proof First, we shall establish the existence and uniqueness of the equilibrium. To do this, assume $\langle \pi^*, 1 \rangle$ is a Nash equilibrium of G, where $\pi^*(q) = med\{s_i\}_{i \in N}$. By way of contradiction, suppose $\langle \tilde{\pi}, \tilde{\omega} \rangle$ is another equilibrium profile. If $\tilde{\omega} = 1$, that is, if the median voter is accepting the agenda-setter's proposal, then $\tilde{\pi}$ must be greater than or equal to π^*. Otherwise, (4.1) implies $u_A(\tilde{\pi}) < u_A(\pi^*)$. At the same time, $\tilde{\pi}$ must be smaller than or equal to s_m, since otherwise m's incentive constraint, $u_m(\tilde{\pi}(q)) \geq u_m(q)$, does not hold. Therefore, $\pi^*(q) \leq \tilde{\pi}(q) \leq s_m(q)$. But Lemma 1 implies $s_m = med\{s_i\}_{i \in N}$. Consequently, $\tilde{\pi} = \pi^*$.

On the other hand, suppose there exists a Nash equilibrium $\langle \tilde{\pi}, \tilde{\omega} \rangle$, such that $\tilde{\omega} = 0$. If $q < \theta_m$, then $\tilde{\omega}$ cannot be optimal, since the setter can then propose $q + \epsilon$, $\epsilon > 0$, which improves his payoff and the utility of the median voter, turning $\tilde{\omega}$ inconsistent with rational choices.[13] Besides, if $q \geq \theta_m$, then $med\{s_i\} = q$. Therefore, the profile cannot be again a Nash equilibrium, since the median voter is indifferent between the agenda-setter's proposal and the reversion point.

The preceding argument show that the strategy profile $\langle \pi^*, 1 \rangle$ is the only possible candidate for a Nash equilibrium of the game. To complete the proof, we show now that $\langle \pi^*, 1 \rangle$ is indeed a Nash equilibrium. To do this, assume $\omega^* = 1$. Then, an optimal strategy for A is $\pi^*(q) = s_m(q)$: for all $\pi < \pi^*$, $u_A(\pi^*) > u_A(\pi)$; moreover, $\pi > \pi^*$ violates m's incentive constraint and therefore will be rejected by the median. In the same way, if $\pi^* = s_m$, then $\omega^* = 1$ is an optimal strategy for m, since by definition s_m makes the median voter indifferent between the agenda-setter's alternative and the reversion point.

Finally, to characterize the equilibrium policy, it is easy to note that (4.2) implies $\pi^* = q$ for all $q \geq \theta_m$. In that case, there is not $x \in X$, $x > q$, such that $u_m(x) \geq u_m(q)$. By the same reasoning, the reader can also check that Euclidean preferences imply $\pi^* = 2\theta_m - q$, for all $q < \theta_m$. □

The following corollary summarizes the main conclusions derived from Proposition 2.

Corollary 1

[13] Note that single-peakedness implies $u_m(q + \epsilon) > u_m(q)$ for all ϵ sufficiently close to zero.

1. $\pi^*(q) \geq \theta_m$, $\forall q \in Q$;

2. $\frac{\partial \pi^*(q)}{\partial q} < 0$, $\forall q < \theta_m$;

3. $\pi^*(q) = q$, $\forall q \geq \theta_m$.

Proof. Trivial. □

Corollary 1 illustrates the main differences between decentralized collective decision-making models and those that allow agenda-setting power. Comparing the equilibrium of the agenda-setter model with the preference-based equilibrium, it follows that the existence of an agenda-setter with strictly increasing preferences induces a policy outcome greater than the median voter's most-preferred policy. Item (1) above captures this idea, by showing that the policy outcome is almost always an over-provision relative to the interests of median voter.

Romer and Rosenthal (1978, 1979a) have shown this difference depends on the status quo location. In particular, (2) shows that lower reversion levels generate higher policies under agenda-setting power, since the status quo threat leads m to accept higher policy proposals. Similarly, (3) points out that reversion levels above m's ideal point give the agenda-setter no additional power to move the policy outcome above the status quo.

4.3 Repetition

The literature on repeated games studies the way in which the future affects the current decisions of the players. In general, this literature shows that the repeated interaction over time produces a great number of new equilibria, by allowing the players to condition their strategies to the way their opponents have played in the past.

In this section, we analyze whether the repetition of the policy-making process of the agenda-setter game could modify the equilibrium strategies of the one-shot version. In order to simplify the analysis, assume that players perfectly observe all previous actions and let $\delta \in (0, 1)$ denote the common discount factor. In addition, suppose the reversion point is given by $q_t = q_0 + \varepsilon_t$, where $\varepsilon_t \in \{\underline{\varepsilon}, \bar{\varepsilon}\}$. In what follows, we assume $\underline{q} \equiv q_0 + \underline{\varepsilon} < \theta_m$ occurs with probability $p \in (0, 1)$ and $\bar{q} \equiv q_0 + \bar{\varepsilon} \geq \theta_m$ with probability $1 - p$.

The game in which the stage game G is played infinitely, G_∞, has a trivial subgame perfect equilibrium, where the equilibrium strategies of G are played in every stage. This is the only equilibrium in which the play at each stage does not depend on the actions previously taken. However, even in this case where the stage game has a unique equilibrium, G_∞ may have many others. As the *Folk Theorem* asserts, any individually rational profile of payoffs can be supported in a Nash equilibrium for a discount factor sufficiently close to 1. Thus,

Proposition 3 *If* $\theta_m > p\,\underline{q} + (1-p)\,\bar{q}$ *and* δ *is close to 1, the result of the Median Voter Theorem remains as a feasible Nash equilibrium of* $G_\infty(\delta)$*,*

Proof First, it must be noted that $(\theta_m,\, u_m(\theta_m))$ is feasible, since for any q, $\langle \theta_m,\, \omega(q,\, \theta_m)\rangle \in \Pi \times \Omega$.[14] Moreover, the minimax of A and m are, respectively, $p\,\underline{q} + (1-p)\,\bar{q}$ and $p\,u_m(\underline{q}) + (1-p)\,u_m(\bar{q})$. Since u_m is single-peaked and $\underline{q} < \theta_m \leq \bar{q}$, $u_m(\theta_m) > p\,u_m(\underline{q}) + (1-p)\,u_m(\bar{q})$. Therefore, if $\theta_m > p\,\underline{q} + (1-p)\,\bar{q}$, the median ideal policy satisfies Folk Theorem's conditions and it constitutes a feasible Nash equilibrium of G_∞. □

Now consider the following *trigger strategy* for player m: accept the agenda setter's proposal in the first period and continue to accepting in every subsequent period τ if $\pi_t(q_t) \leq \theta_m$ for all $t \leq \tau$, and reject for the rest of the game if $\pi_t(q_t) > \theta_m$ for some $t = 2, 3, ..., \tau$. Given this trigger strategy, the optimal strategy for A is as follows. First, notice that the agenda setter expected payoff is,

$$\frac{\theta_m}{1-\delta} \tag{4.12}$$

if he proposes $\pi_t(q_t) = \theta_m$ in every period; and,

$$\pi_1(q_1) + \frac{\delta\,[p\,\underline{q} + (1-p)\,\bar{q}]}{1-\delta} \tag{4.13}$$

if he deviates in some stage, says period one, choosing $\pi_1(q_1) > \theta_m$.[15] Then, A's proposal will be equal to the median voter's most-preferred alternative whenever,

$$\frac{\theta_m}{1-\delta} > \pi_1(q_1) + \frac{\delta\,[p\,\underline{q} + (1-p)\,\bar{q}]}{1-\delta} \tag{4.14}$$

which requires,

$$p\,\underline{q} + (1-p)\,\bar{q} < \frac{\theta_m - (1-\delta)\pi_1}{\delta} \tag{4.15}$$

Thus, if δ is close to one, (4.15) holds if $\theta_m > p\,\underline{q} + (1-p)\,\bar{q}$

[14] The set of feasible payoffs is defined as $\{(v_m(\pi,\, \omega),\, v_A(\pi,\, \omega)) : (\pi,\, \omega) \in \Pi \times \Omega\}$.

[15] In order to determine which flow of payoffs is greater, the case in which A deviates in some other period different from one can be easily transformed into the case in which A chooses $\pi_1 > \theta_m$.

This result is consistent with Proposition 3, and it suggests the following. If the discount factor is close enough to one, which means that the future is important for the players, and m's ideal point is greater than the expected payoff obtained in the case of rejection, then the agenda-setter has not incentives to deviate from cooperation. Even though he could do better in the short-run by defeating, for sufficiently patient players any finite one-period gain from deviation is outweighed by even small losses in every future period.

But, is the trigger strategy an optimal strategy for m? A player could have the capacity to punish other agents, but such punishment could be very costly for him. In such case, he could prefer not to implement it. In other words, these punishing strategies could not necessarily conduce to a subgame perfect equilibrium.

Fortunately, perfect Folk Theorem guarantees that, for any feasible individually rational profile of payoffs, there is a range for the discount factor for which those payoffs can be supported in a subgame perfect equilibrium of the infinitely repeated game.

Suppose players evaluate sequences of per-period utilities by the time-average criterion.[16]

Remark 1 *If $\theta_m > p\,\underline{q}+(1-p)\,\bar{q}$, then there is a subgame perfect equilibrium in $G_\infty(\delta)$ that supports the median voter's most-preferred alternative as the policy outcome.*

PROOF Suppose m uses the following strategy: accept the agenda setter's proposal in the first period and continue to accepting in every subsequent period τ if $\pi_t(q_t) \leq \theta_m\ \forall t \leq \tau$ and reject for n periods if $\pi_t(q_t) > \theta_m$ for some $t = 2, 3, \ldots, \tau$. Assume also that n is chosen so that,

$$p\,[s(\underline{q}) - \underline{q}] < n\,\{\theta_m - p\,\underline{q} + (1-p)\,\bar{q}\}$$

Then, if $\theta_m > p\,\underline{q}+(1-p)\,\bar{q}$, this condition on n ensures that any gain from deviation in the cooperative phase is removed at the punishment phase, so that no sequence of finite or infinite number of deviations can increase player A's average payoff above θ_m. Moreover, even though minimaxing a deviator is costly in terms of per-period payoffs, any finite number of such losses is costless with the time average criterion. In particular, m's expected payoff

[16] In fact, it is not necessary that players use the time-average criterion, which assumes no discounting at all. As Fudenberg and Maskin (1986) have shown, in two-player repeated games the set of perfect equilibrium outcomes always converges to the individually rational set of payoffs as the discount factor tends to one. Thus, an exact counterpart of Proposition 3 could be established, by replacing the word "Nash" by "perfect".

if agent A deviates γ times,

$$\frac{\gamma\, n\{p\, u_m(\underline{q}) + (1-p)\, u_m(\bar{q})\} + (T-\gamma)\, u_m(\theta_m)}{T}$$

tends to $u_m(\theta_m)$ as $T \to \infty$. This means that the median voter's average payoff in a subgame where the agenda-setter is being punished is $u_m(\theta_m)$, so that he gains nothing by deviating. □

Remark 1 ensures the existence of a perfect equilibrium profile of strategies that leads the outcome of the agenda-setter model to the median voter's most-preferred alternative. This result preserves the relevance of the Median Voter Theorem in repeated one-dimensional collective decision games, and suggests that Romer and Rosenthal's main conclusion could not necessary holds if agents are allowed to play the same game many times.

Finally, it is important to note that an implicit assumption essential for our results is the impossibility of agents to negotiate away from bad equilibria. A difficulty arises when agents can negotiate away from bad outcomes, as the power of threats is weakened and this, in turn, reduces the scope for sustaining good outcomes. If cheating occurs and the equilibrium specifies punishments that are detrimental to all agents, individuals should have incentives to propose mutually preferable outcomes. In our case, notice that the punishment strategy does not satisfy the renegotiation-proof criterion, since the payoffs corresponding to the cooperative phase Pareto dominate those of the punishment phase, meaning that the equilibrium induced by this strategy profile is not an efficient outcome and, therefore, that agents as a group could improve by moving to other equilibrium.

4.4 Conclusion

This chapter formalizes the agenda-setter model and extends it to a dynamic environment. The benchmark game, represented by two long-run players, the agenda-setter and a representative voter, suggests the existence of a unique subgame perfect equilibrium, with the same properties found by Romer and Rosenthal. In particular, it shows that an agenda-setter with strictly increasing preferences may induce a policy outcome greater than the median voter's most-preferred alternative and that this policy depends on the status quo location.

The second part introduces repetition into the benchmark game. Assuming an infinity horizon and a common discount factor close enough to one, Proposition 3 and Remark 1 points out that, if the representative voter's ideal point is greater than the expected payoff obtained under rejection, then the median ideal point can be sustained in a subgame perfect equilibrium. This implies that the institutional bias found by Romer and Rosenthal can be removed if agents are allowed to interact over time. In particular, it shows that this will be case if the median voter plays a strategy that begins cooperating and then changes to a finite punishment phase after observing any deviation of the agenda-setter.

Chapter 5

Separation of powers and political budget cycles

5.1 Introduction

The literature on political budget cycles (henceforth PBC) studies cycles in fiscal policies generated by the electoral process. There may be cycles in the size of the budget, in the composition of public spending, and in the choice of taxes or debt to finance expenditures.

Though at the theoretical level this literature has made significant progress, an analysis of PBC under separation of powers remains to be done.[1] In effect, none of the existing models of rational PBC has incorporated the legislature as a second policy-maker in the decision-making process followed to set fiscal policy. That is, in all these models it is implicitly assumed that fiscal decisions are taken unilaterally by the executive, without any kind of institutional constraints. This paper formally tackles the impact of separation of powers on fiscal policy distortions in a specific model of PBC in the composition of government spending. To the best of our knowledge of the field, this is the first time such goal is carried out.[2]

Separation of powers brings into play a system of checks and balances. In this regard, in all constitutional democracies a relatively fixed and well-known procedure is followed every year to determine the annual budget of expenditures and the public resources to finance it. This paper attempts to explicitly analyze the role of such bargaining process, by which the budgetary

[1] See Shi and Svensson (2003) for a recent review.

[2] In the analysis of the electoral cycle in monetary policy, Lohmann (1998a) and Drazen (2001) consider the interplay of two policy-makers, modeling a fiscal authority (represented by the executive office) and a monetary authority (the central bank).

law is drafted, approved and implemented.

There are also empirical reasons for this study. One motivation is the literature on budget procedures and budget institutions, which points to their significant impact on fiscal outcomes, and their role in explaining the cross-country variance of fiscal experiences.[3] Thus, one could also expect budget institutions significantly influence fiscal policy at electoral dates.

Another motivation is the fact that in many empirical papers on PBC, political institutions have a significant impact over pre-electoral and post-electoral distortions of fiscal policy. Effectively, recent empirical research suggests that fiscal policy tends to be systematically manipulated before elections, reporting evidence on changes in taxes, deficit, aggregate spending and spending composition. Moreover, they find these electoral cycles to be more pronounced in developing countries, ruled in most cases by worse democratic institutions. While not focusing explicitly on budget institutions, this empirical evidence constitutes a further motivation for this study.[4]

In order to explain some of these empirical regularities and to fill out the theoretical gap, this paper explicitly analyzes the effect of the electoral calendar on the composition of government spending under separation of powers. As in Rogoff and Sibert (1988), Rogoff (1990) and others, the incumbent faces before elections an incentive to boost the supply of the more visible (consumption) public goods, in the hope that voters will attribute the boost to its competence and will reelect it for another term. However, instead of assuming an all-powerful executive, our model introduces a legislature into the decision-making process, reflecting in a stylized way the mechanism by which the annual budget of expenditures is drafted, approved and implemented.

The main result shows that effective checks and balances in the budgetary process curb PBC. Concretely, it shows that the institutional features of the legislative bargaining game, namely, the actual agenda-setting authority, the status quo location and the degree of the legislative oversight of the implementation of the budgetary law, play critical roles for explaining the existence and magnitude of electoral cycles in fiscal policy.

The chapter is organized as follows. Section 5.2 presents the model. The equilibrium analysis is carried out in section 5.3. Finally, section 5.4 outlines the main results and directions for future research.

[3]See Alesina and Perotti (1995) for a survey of this literature. For empirical works, see for example Alesina et al. (1999) and Woo (2003), and the references quoted there.

[4]For further details, see, among others, Block (2002), González (2002), Persson and Tabellini (2002), Shi and Svensson (2002a, 2002b, 2003) and Schuknecht (1998).

5.2 The model

Consider an infinite-horizon society composed by a large but finite number of identical individuals, labeled $i = 1, 2, \ldots, n$. Let t denote time, $t \in T \equiv T_1 \cup T_2$, where T_1 is the set of all odd positive integers (electoral periods) and T_2 is the set of all even positive integers (post-electoral periods).

In every period $t \in T$, individual i plays roles both as a consumer and as a citizen. The representative consumer derives utility from two types of public goods, which differ in the timing of their production: a consumption good $g_t \in \Re_+$, instantaneously supplied and immediately visible, and a capital good $k_{t+1} \in \Re_+$, provided at the end of period t. The capital good cannot be observed until it is in place.

To simplify the equilibrium characterization, it is assumed that the representative consumer's per-period payoff is given by a Cobb-Douglas utility function $u : \Re_+ \times \Re_+ \rightarrow \Re$,

$$u(g_t,\, k_{t+1}) = (g_t)^{\alpha}\; (k_{t+1})^{1-\alpha}, \tag{5.1}$$

where $0 < \alpha < 1$.

In each period $t \in T$, the economy is subject to the budget constraint

$$\gamma_t + \kappa_t = \tau, \tag{5.2}$$

where γ_t, $\kappa_t \in \Re_+$ denote *actual budget expenditures* on consumption and capital goods, respectively, and $\tau \in \Re_{++}$ is a fixed sum of tax revenues (the size of the public sector).

The production of public goods is such that the same amount of per-period public resources can be transformed into either one unit of g_t or one unit of k_{t+1}. Their *effective provision* is affected by a random variable, $\theta_t > 0$, that represents the *competence* of the agent in charge of this task (to be specified below). It is determined by,

$$g_t = \theta_t\, \gamma_t \tag{5.3}$$

$$k_{t+1} = \theta_t\, \kappa_t. \tag{5.4}$$

Actual competence is assumed to be partially lasting, following a first-order moving average (MA(1)) process, as in Rogoff and Sibert (1988), Rogoff (1990) and others,

$$\theta_t = \bar{\theta} + \varepsilon_t + \varepsilon_{t-1}, \tag{5.5}$$

where ε is a random *iid* variable and ε_t denotes the period t realization of ε. Our interpretation of these competence shocks is that, although competence is in principle persistent, it comprises multiple dimensions that are not necessarily correlated. The specific challenges a government faces change exogenously over time, making *actual* competence contingent to these changes.

The variable ε is uniformly distributed over the interval $\left[-\frac{1}{2\xi}, \frac{1}{2\xi}\right]$, with expected value $E(\varepsilon) = 0$ and density function $\xi > 0$. A higher value of ε corresponds to a more competent politician, since the same per-period tax revenues can be used to provide more of both public goods. The marginal (conditional on ε_{t-1}) probability distribution of θ_t, $\tilde{F}(\cdot) = F(\cdot\,|\varepsilon_{t-1})$, is also uniform, with support $\Theta_t = \left[\bar{\theta} + \varepsilon_{t-1} - \frac{1}{2\xi}, \bar{\theta} + \varepsilon_{t-1} + \frac{1}{2\xi}\right]$, $\tilde{F}' > 0$ for all $\theta_t \in \Theta_t$, and $E(\theta_t) = \bar{\theta} + \varepsilon_{t-1}$. Henceforth, it is assumed that $\bar{\theta} > 1/\xi$, so that θ_t is always positive and (3) and (4) are well-defined.

5.2.1 Separation of powers

In contrast to much of the theoretical literature on PBC, in this work the policy-making process carried out to set the mix of public expenditures involves the interaction of two political agents, labeled E and L. These agents are the current leaders, or *incumbents*, of the two branches of government: the *executive* and the *legislature*.

In each branch, a leader's term lasts two periods. Every other period, a random *iid recognition rule* $\hat{L} : T_1 \rightarrow \{1, 2 \ldots, n\}$ selects a new leader for the legislature from the set of all possible political candidates, which coincides here with the set of citizens.[5] On the other hand, the electorate removes or confirms the executive leader in an explicit electoral contest. If the executive incumbent is confirmed, it controls this branch for another term. Otherwise, a new policy-maker is randomly recognized according to the rule $\hat{E} : T_1 \rightarrow \{1, 2, \ldots, n\}$. Except where indicated, no limit is set on the number of times incumbents can run for reelection.

Incumbents' per-period payoffs are as follows. They receive, like other citizens, utility from the consumption of public goods, and they also receive an exogenous rent $\chi > 0$, reflecting the satisfaction from being in power. These rents will be the source of conflict between the incumbents and the electorate. In Lohmann's (1998b) words, this variable reflects the strength of the electoral goal.

In each period $t \in T$, incumbents observe all variables, except the value of ε_t, before making budget allocation decisions. That is, they choose and implement policies before the executive leader's competence is realized. This assumption simply implies that, *ex-ante*, incumbents are uncertain about

[5] To simplify the analysis, neither the legislative electoral process nor the citizens' individual decision of entering into the political arena are modeled.

how well they will be able to transform government revenues into public output.

On the other hand, the electorate does not observe the executive leader's most recent competence shock, ε_t, nor the allocation (γ_t, κ_t) and k_{t+1} before voting. The only information it receives is the amount of the consumption good, g_t, that is provided. Thus, incumbents have a temporary information advantage over the actual budget allocation implemented. The electorate does not observe it until the end of each period. All past competence shocks, as well as the amount of tax revenues, are common knowledge. Finally, even though voters do not observe the particular circumstances incumbents confront at a given date, they know the incentives they face and the objectives they try to achieve.

5.2.2 Checks and balances

The process for setting the budgetary mix under separation of powers involves a specific system of *"checks and balances"* among them. At the stage of budget formulation and approval, the institutional arrangement gives E the right to make a budget allocation proposal, but it requires the motion to be accepted by L. If no amendment rights exist, L faces a take-it-or-leave-it proposal, where the reversion outcome (the status quo) in case of rejection is exogenously specified. The legislature might be allowed to amend the executive's proposal, but then the amended proposal can be vetoed by E.[6]

At the implementation stage, the executive supplies the public goods, but it can be monitored to a certain extent by the legislature. Specifically, it is assumed that an exogenous proportion $\delta \in [0, 1]$ of $\tilde{\kappa}_t$, the expenditures approved for the provision of the public capital good, cannot be reassigned.[7] The interpretation is that these resources represent public funds affected to specific ends, whose realizations are subject to the revision of the legislature. Thus, at the implementation stage, the executive's leader can at most reassign an amount $(1 - \delta)\,\tilde{\kappa}_t$ of resources to the provision of g_t. The measure δ determines the effective limits the legislature imposes on the executive office.[8]

[6]The possibility that L overrides E's veto, not considered here, is trivial to analyze. However, this is an unlikely case, since it usually requires that the majority leader L in the legislature have a *qualified* majority to impose its criterion when E vetoes an amendment.

[7]It will be clear below that incumbents confronted with electoral contests refrain from transferring resources from g_t to k_{t+1}. The reason is only the provision of the more *visible* consumption goods will be effective for the incumbents' purpose of appearing talented to voters before elections.

[8]Notice that the legislature is endowed only with the power to guarantee some specific items will be supplied, but not to prevent the over-provision of certain public goods.

5.2.3 The game

Given the MA(1) process for competence, the infinite-horizon model described above can be broken down into a sequence of two-period sequential games, in which each election is independently analyzed. Consider one of these games, which will be referred to as G. Call t and $t+1$ its two periods, such that $t \in T_1$ and $t+1 \in T_2$. The set of players of G are the two incumbents, E and L, the *representative voter*, V, and Nature.[9]

Let $\Gamma = [0,\, \tau] \subset \Re_+$ be the set of feasible expenditures on the public consumption good. A pure strategy for E in G is a pair $\lambda^E = (\lambda^E_t,\, \lambda^E_{t+1})$ such that, for each $s \in \{t,\ t+1\}$, $\lambda^E_s = (\tilde{\gamma}^E_s,\, d^E_s,\, \gamma_s)$, where [10]

- $\tilde{\gamma}^E_s : \Gamma \rightarrow \Gamma$ is the *budget allocation proposal*, as a function of the status quo $\bar{\gamma} \in \Gamma$;
- $d^E_s : \{0,\, 1\} \times \Gamma \times \Gamma \rightarrow \{0,1\}$ is a *veto decision rule*, which depends on L's approval or rejection of $\tilde{\gamma}^E_s$, the *amended proposal* $\tilde{\gamma}^L_s$ in case of rejection (to be specified below) and $\bar{\gamma}$; and,
- $\gamma_s : \Gamma \times [0,1] \rightarrow \Gamma$ denotes *actual expenditures* on g_s, which depends on $\delta \in [0,1]$ and *authorized expenditures* $\tilde{\gamma}_s \in \Gamma$ (yet to be defined).

In the same way, a pure strategy for L in G is a pair $\lambda^L = (\lambda^L_t,\, \lambda^L_{t+1})$ such that, for each $s \in \{t,\ t+1\}$, $\lambda^L_s = (d^L_s, \tilde{\gamma}^L_s)$, where

- $d^L_s : \Gamma \times \Gamma \rightarrow \{0,1\}$ is a *veto decision rule*, given $\tilde{\gamma}^E_s$ and $\bar{\gamma}$; and,
- $\tilde{\gamma}^L_s : \Gamma \times [0,1] \rightarrow \Gamma(\tilde{\gamma}^E)$ is an *amendment rule*, as a function of $\bar{\gamma}$ and δ, where $\Gamma(\tilde{\gamma}^E) \subseteq \Gamma \cup \{\emptyset\}$ represents the set of feasible amendments to $\tilde{\gamma}^E_s$. For simplicity, it is assumed that $\Gamma(\tilde{\gamma}^E)$ does not depends on $\tilde{\gamma}^E_s$. The analysis will focus on two extreme cases: (i) *Closed rule:* $\Gamma(\tilde{\gamma}^E) = \emptyset$ and (ii) *Open rule:* $\Gamma(\tilde{\gamma}^E) = \Gamma$.

[9]Two comments are in order. First, since individuals are identical, there is no loss of generality in using a single representative voter. Second, the two potential incumbents $\hat{E}(t)$ and $\hat{L}(t)$ should formally be included in the set of players. However, since these players (potentially) participate only in the last period of the game, and the optimal strategies of all incumbents at this post-electoral period are the same, the distinction between them and the original incumbents will be omitted. This simplifies the notation considerably.

[10]In an abuse of notation, λ^E_{t+1} is used to denote both a (conditional on being reelected) strategy for E at $t+1$ and a possible plan of action for the (potential) executive incumbent $\hat{E}(t)$. This simplification is also adopted below for L's strategies. Notice that it entails no loss of generality because all incumbents choose the same optimal strategy in the last period of the game.

Finally, in order to decide its vote, it is assumed that V compares the flow of payoffs expected under each of the potential executive incumbents. That is, in the electoral period t, it behaves according to the forward-looking *voting rule* $\lambda^V : \Re \rightarrow \{0, 1\}$,[11]

$$\lambda^V = \begin{cases} 1 & \text{if } E_t\{v(\gamma_{t+1},\, \theta_{t+1}) \mid E\} \geq E_t\{v(\gamma_{t+1},\, \theta_{t+1}) \mid \hat{E}(t)\}, \\ 0 & \text{otherwise.} \end{cases} \tag{5.6}$$

where, for each $s \in \{t,\, t+1\}$, $v(\gamma_s,\, \theta_s) \equiv u[\theta_s \gamma_s,\; \theta_s(\tau - \gamma_s)]$ is the indirect utility function. $\lambda^V = 1$ represents V's decision to keep the current executive incumbent in office, while $\lambda^V = 0$ is vote to replace it.

For each $j \in \{E,\, L,\, V\}$, let Λ^j denote player j's set of pure strategies. A pure strategy profile in G is a vector $\lambda = (\lambda^E,\, \lambda^L,\, \lambda^V) \in \Lambda$, where $\Lambda \equiv \prod_j \Lambda^j$. Then, for each $j \in \{E,\, L,\, V\}$, player j's payoffs in G are given by a mapping $\pi^j : \Lambda \rightarrow \Re$, such that:

$$\pi^E(\lambda) = E_t \left\{ \sum_{s=t}^{t+1} \beta^{s-t}\; v(\gamma_s,\, \theta_s) + \mu_s^E\, \chi \,\Big|\, \lambda \right\}, \tag{5.7}$$

$$\pi^L(\lambda) = E_t \left\{ \sum_{s=t}^{t+1} \beta^{s-t}\; v(\gamma_s,\, \theta_s) + \mu_s^L\, \chi \,\Big|\, \lambda \right\}, \tag{5.8}$$

$$\pi^V(\lambda) = E_t \left\{ \sum_{s=t}^{t+1} \beta^{s-t}\; v(\gamma_s,\, \theta_s) \,\Big|\, \lambda \right\}, \tag{5.9}$$

where $\beta \in (0, 1)$ is a common discount factor and μ_s^j is the probability incumbent j attaches to being in office in period s,

$$\mu_s^E = \begin{cases} 1 & \text{if } s = t, \\ \text{prob}(\lambda^V = 1 \mid \lambda_{s-1}^E,\, \lambda_{s-1}^L) & \text{if } s = t+1, \end{cases} \tag{5.10}$$

and

$$\mu_s^L = \begin{cases} 1 & \text{if } s = t, \\ \text{prob}(\hat{L}(s-1) = L) & \text{if } s = t+1. \end{cases} \tag{5.11}$$

In each period $s \in \{t,\, t+1\}$, the timing of events is as follows:

1. E submits $\tilde{\gamma}_s^E$ to L;

2. L receives $\tilde{\gamma}_s^E$ and

[11] The restriction of λ^V to pure strategies (to a *yes* or *no* vote) makes sense in large populations, since it may be unrealistic to assume that voters coordinate on implementing a strategy that makes reelection random from the point of view of the executive incumbent.

(i) If $\Gamma(\tilde{\gamma}^E) = \emptyset$, L chooses whether to accept $\tilde{\gamma}_s^E$ or not; and

$$\tilde{\gamma}_s = \begin{cases} \tilde{\gamma}_s^E & \text{if } d_s^L = 1, \\ \bar{\gamma} & \text{if } d_s^L = 0; \end{cases} \tag{5.12}$$

(ii) If $\Gamma(\tilde{\gamma}^E) = \Gamma$, L decides whether to amend $\tilde{\gamma}_s^E$ or not; if it is modified, E chooses whether to veto $\tilde{\gamma}_s^L$ or not; and

$$\tilde{\gamma}_s = \begin{cases} \tilde{\gamma}_s^E & \text{if } d_s^L = 1, \\ \tilde{\gamma}_s^L & \text{if } d_s^L = 0 \text{ and } d_s^E = 1, \\ \bar{\gamma} & \text{if } d_s^L = 0 \text{ and } d_s^E = 0; \end{cases} \tag{5.13}$$

3. E implements γ_s, which may differ from plan $\tilde{\gamma}_s$ if $\delta < 1$;

4. ε_s is realized and g_s and k_{s+1} are determined according to (5.3) and (5.4);

5. V observes g_s, but not k_{s+1}, nor ε_s and (γ_s, κ_s);

6. If $s = t$, $\hat{L}$ chooses a new legislative leader for the next political term. Simultaneously, V decides whether to vote for E or not. If E is reelected, it stays in office for two more periods. If not, $\hat{E}$ chooses a new executive leader, whose competence at $t+1$ is determined by Nature from the probability distribution of ε;

7. Individuals observe k_{s+1} and period s ends.

Since this game is not of perfect information, the equilibrium concept used to solve it is (pure strategies) weak perfect Bayesian equilibrium. This equilibrium concept involves an explicit description of players' beliefs, which must be statistically consistent with the strategy profile, as well as the optimality requirement that agents must choose a best response to the other players' strategies, given their beliefs. More precisely,

Definition: *A pure strategy equilibrium for G is a profile of strategies $\hat{\lambda} = (\hat{\lambda}^E, \hat{\lambda}^L, \hat{\lambda}^V)$ and a belief for competence θ_t^e such that, in any continuation game of G,*

- *Given $\hat{\lambda}^{-j}$ and the specified belief θ_t^e, each player $j \in \{E, L, V\}$ weakly prefers $\hat{\lambda}^j$ to λ^j, for all $\lambda^j \in \Lambda^j$;*

- $\theta_t^e \equiv E_t[\theta_t \mid g_t]$ *is determined using Bayes rule and* $\hat{\lambda}$ *on the equilibrium path; off the equilibrium path, it is determined by the condition that unexpectedly low values of* g_t *correspond to minimum competence, while unexpectedly high values of* g_t *correspond to maximum competence.*

5.3 Equilibrium analysis

This section analyzes the effects of different institutional arrangements over the size of the electoral cycle in the composition of public expenditures. It starts by considering the case without elections.

5.3.1 Benchmark

Suppose no electoral contest is held. That is, assume a unique individual is randomly selected at the beginning of period t, after which it controls both the executive and the legislature. Let $\Delta = |\gamma_{t+1} - \gamma_t|$ denote the size of the electoral cycle on budget expenditures γ.

Remark 1 *If there are no elections, then every period equilibrium expenditures are given by* $\gamma^* = \alpha\ \tau$ *and* $\kappa^* = (1-\alpha)\ \tau$*. Hence, cycle* $\Delta^* = 0$*.*

This is the social planner's solution, which is obtained in the usual way. Since this result is familiar to most readers, the proof is omitted.

5.3.2 One policy-maker

Assume now an electoral contest takes place every other period. One can assume that only one policy-maker I $(= E = L)$ exists, or that the result of the legislative electoral process, represented by $\hat{L}$, is perfectly correlated with the outcome of the presidential election. This is the case of unified government, which is the usual situation analyzed in the literature on rational PBC. The superscript u will stand for equilibrium values under "unified government".

Proposition 1 *If there is unified government, then there exists a unique pure strategy equilibrium* $\hat{\lambda}^u$ *in* G *such that* $\gamma_{t+1}^u = \gamma^*$*,* $\gamma_t^u > \gamma^*$ *is implicitly defined by the condition*

$$\left(\frac{\gamma_t^u}{\tau - \gamma_t^u}\right)^{\alpha} (\gamma_t^u - \alpha\tau) = \beta\,\chi\,\tilde{F}\,',$$

and $\lambda^V = 1$ *if and only if* $\theta_t^e = g_t/\gamma_t^e \geq \bar{\theta}$*.*

Corollary 1 *There is electoral cycle, where*

1. $\Delta^u > 0$;
2. Δ^u *is strictly increasing in both* χ *and* β.

Thus, this simple model predicts optimal equilibrium policy during off-electoral periods, but not during electoral periods. These specific results are pretty standard, having to do with the MA(1) nature of competency shocks (cf. Appendix for proofs).

The intuition is that in post-electoral periods there is no incentive to distort policy, since reputation of competence only lasts one period. However, in pre-electoral periods there is an incentive to distort the composition of government spending. Competent incumbents are reelected, whereas incompetent ones are removed from office. Hence, the incumbent's incentive to appear competent induces overspending on the public consumption good (the more visible good), at the expense of the public capital good (the less visible good). The incumbent trades-off the distortions in the composition of public expenditures (about which it cares) against a higher probability of winning the electoral contest.

Even though the policy bias in electoral periods reduces voters' welfare, there is a positive selection effect because elections help to select candidates with above-average competency for office (the net effect may be positive or negative, cf. Lohmann 1998b).

Why is it that the optimal allocation at date t cannot be sustained in equilibrium? Since I does not observe ε_t before the election, g_t cannot provide any useful information about its most recent competence shock. Thus, both V and I would be better off with budget allocation γ^* instead of γ^u. However, I cannot credible compromise to follow γ^* during electoral periods. If such policy were expected by V, then I would have an incentive to exploit its discretionary power to deviate to γ^u, since such deviation would increase its probability of being reelected. Hence this cannot be part of an equilibrium.

In PBC models à la Lohmann, where policy choices are made before the competence shock is realized, the credibility problem depicted above is at the heart of the electoral distortion of the fiscal policy.[12] As the next section will

[12] In signaling PBC models à la Rogoff, the incumbent observes its competence before choosing the per-period policy. Therefore, the alternative chosen signals to the uninformed player valuable information about the incumbent's competence. Though the electoral distortion on policy outcomes is due to information transmission, not only to lack of credibility, this does not mean that the informativeness of the signal is larger in equilibrium: with both the Rogoff and the Lohmann timing, there is a separating equilibrium. Besides, the Rogoff timing brings in extra complications that are not required to explain the policy bias in electoral periods.

show, it turns out that this problem is in fact generated by concentration of powers, which allows E to choose any policy it desires. Instead, when there exists more than one policy-maker, separation of powers, by requiring joint agreement in decision-making, restricts its capacity of unilateral moves and therefore works as a commitment device that makes all players better off (including the executive incumbent). This policy bias in electoral periods is comparable, for example, to the credibility problem in the Barro-Gordon model, where the inflation bias can be solved through institutional solutions like the Central Bank independence and conservative central bankers.

5.3.3 Two policy-makers

This section incorporates a second policy-maker, the legislature, into the model, as well as the institutional structure of checks and balances discussed in section 5.2. The main purpose is to analyze how the results of the previous section change after these modifications are introduced.

Two cases have to be considered, depending on whether $\Gamma(\tilde{\gamma}^E) = \emptyset$ or $\Gamma(\tilde{\gamma}^E) = \Gamma$.

(i) Closed rule

Assume no amendments can be made to the executive's proposal. That is, following the jargon of the legislative bargaining literature, suppose there exists a closed rule, so that the legislature faces each period a take-it-or-leave-it allocation proposal, with its rejection followed by the exogenous reversion point $(\bar{\gamma}, \bar{\kappa})$.

For $j \in \{E, L\}$, let $\tilde{\pi}^j(\cdot)$ denote player j's policy preferences over Γ, with ideal policy $\gamma^j = \arg\max_{\gamma} \tilde{\pi}^j(\gamma)$.[13] Define a matching function $r^j : \Gamma \rightarrow \Gamma$ as follows: $\forall\ \gamma' \in [0, \gamma^j]$, set $r^j(\gamma') = \gamma''$ if there exists $\gamma'' \in [\gamma^j, \tau]$ such that $\tilde{\pi}^j(\gamma') = \tilde{\pi}^j(\gamma'')$, and $r^j(\gamma') = \tau$ otherwise. Similarly, $\forall\ \gamma' \in [\gamma^j, \tau]$, fix $r^j(\gamma') = \gamma''$ if there exists $\gamma'' \in [0, \gamma^j]$ such that $\tilde{\pi}^j(\gamma') = \tilde{\pi}^j(\gamma'')$, and $r^j(\gamma') = 0$ otherwise. The superscript s stands for equilibrium values under "separation of powers".

Proposition 2 *Suppose there is separation of powers. If the legislature cannot amend the executive's proposal, then there exists a unique pure strategy*

[13] The ideal policy of each incumbent is the policy it would choose if it were not constrained by the requirement that its proposal has to be approved by the other policy-maker.

equilibrium $\hat{\lambda}^s$ *in* G *such that* $d_{t+1}^{L,s} = d_t^{L,s} = 1$, $\tilde{\gamma}_{t+1}^{E,s} = \gamma_{t+1}^{s} = \gamma^*$,

$$\tilde{\gamma}_t^{E,s} = \begin{cases} \max\left\{\bar{\gamma},\ r^L(\bar{\gamma})\right\} & \text{if } \bar{\gamma} \in \left(r^L(\gamma^E),\, \gamma^E\right), \\ \gamma_t^u & \text{otherwise,} \end{cases} \tag{5.14}$$

$$\gamma_t^s = \min\left\{\gamma_t^u,\ \tau - \delta[\tau - \tilde{\gamma}_t^s]\right\}, \tag{5.15}$$

and $\lambda^V = 1$ *if and only if* $\theta_t^e = g_t/\gamma_t^e \geq \bar{\theta}$.

Let $\delta^{crit}(\bar{\gamma}) \equiv \frac{\tau-\gamma^u}{\tau-\tilde{\gamma}^s(\bar{\gamma})}$ be the critical level of discretion that makes the first term of the right hand side of (5.15) equal to the second.

Corollary 2 *Electoral cycles depend on the status quo and overview power of legislature,*

1. *If* $\bar{\gamma} \in \left(r^L(\gamma^E),\, \gamma^E\right)$ *and* $\delta > \delta^{crit}(\bar{\gamma})$, *then* $\Delta^* \leq \Delta^s(\bar{\gamma},\, \delta) < \Delta^u$;[14]

2. *If either* $\bar{\gamma} \in \left[0,\, r^L(\gamma^E)\right] \cup \left[\gamma^E,\, \tau\right]$ *or* $\delta \leq \delta^{crit}(\bar{\gamma})$, *then* $\Delta^s(\bar{\gamma},\, \delta) = \Delta^u$;

3. *Given* $\bar{\gamma} \in [0,\, \tau]$, $\Delta^s(\bar{\gamma},\, \delta)$ *is non-increasing in* δ.

To derive Proposition 2, consider first the post-electoral period $t+1$. Following the same argument applied in section 5.3.2, it is immediate to note that the incumbents implement their common $t+1$-most-preferred policy γ^*. No agent can be made better off by unilateral deviations.

Going back to the electoral period t, the problem for V is still to estimate the competence of E, $E_t[\theta_t|\, E]$, after having observed g_t. As in the previous section, for the expected equilibrium policy γ_t^e, $\theta_t^e = g_t/\gamma_t^e$. Therefore, μ_{t+1}^E has the same form that (5.20).[15]

However, γ_t is now determined in a non-trivial bargaining process between the executive and the legislature, instead of being unilaterally set by E. Under the closed rule, E has maximum power in the bargaining game. Therefore, it can be conjectured that L will be nailed to its status quo payoff. Based on this conjecture, the process is solved in the following way. Consider first incumbents' preferences over γ_t. For each $j \in \{E,\, L\}$, let $\tilde{\pi}^j : \Gamma \to \Re$ denote player j's payoff as a function of γ_t:

$$\begin{aligned} \tilde{\pi}^j(\gamma_t) &= \pi^j(\gamma_t \,|\, \tilde{\gamma}_t^E,\, d_t^L,\, \lambda^V,\, \hat{\lambda}_{t+1}^s), \\ &= E_t\left\{v(\gamma_t,\, \theta_t) + \chi + \beta\left[v(\gamma^*,\, \theta_{t+1}) + \mu_{t+1}^j\, \chi\right] \Big|\, \tilde{\gamma}_t^E,\, d_t^L,\, \lambda^V\right\}. \end{aligned}$$

[14] Notice that $\Delta^s(\bar{\gamma},\, \delta) = \Delta^*$ only if $\bar{\gamma} = \gamma^*$.

[15] Here, it is assumed that V knows both $\bar{\gamma}$ and δ. However, the qualitative results do not change if these variables are not observed by V. In that case, γ_t^e will be a function of the estimated values of $\bar{\gamma}$ and δ, but the inference process in (5.19) is the same.

It is immediate to see that $\tilde{\pi}^j$ is single-peaked on Γ, with ideal policies $\gamma^L = \gamma^*$ and $\gamma^E = \gamma^u$.[16]

Figure 5.1: Incumbents' preferences over γ_t.

In order to pass a proposal $\tilde{\gamma}_t^E$, E has to guarantee L at least its reservation payoff $\tilde{\pi}^L(\bar{\gamma})$, to persuade it not to reject $\tilde{\gamma}_t^E$. That is, the executive's proposal has to satisfy the incentive constraint

$$\tilde{\pi}^L(\tilde{\gamma}_t^E) \geq \tilde{\pi}^L(\bar{\gamma}). \tag{5.16}$$

Therefore, the problem of E at date t is to choose $\tilde{\gamma}_t^E$ in order to maximize $\tilde{\pi}^E(\gamma_t)$ subject to (5.16) and (5.20). Looking at Figure 5.1, it is clear that only two cases are possible. If $\bar{\gamma} \in [0,\, r^L(\gamma^u)] \cup [\gamma^u,\, \tau]$, then (5.16) is not binding, since $\tilde{\pi}^L(\gamma^u) > \tilde{\pi}^L(\bar{\gamma})$ for all $\bar{\gamma} \neq \gamma^u$. That is, the reversion outcome is too low or too high, so that L is unable to affect the equilibrium budgetary policy $\tilde{\gamma}_t^s$, by triggering E to refuse its proposal. V anticipates this and expects E will obtain in equilibrium authorized expenditures $\tilde{\gamma}_t^s = \gamma^u$. Therefore, the same reasoning of section 5.3.2 applies.

On the other hand, if $\bar{\gamma} \in (r^L(\gamma^u),\, \gamma^u)$, then $\tilde{\gamma}_t^s$ will be above γ^*, but below γ^u. Concretely, since L would reject any other proposal that violates (5.16), E ties L to its status quo payoff, by proposing $\tilde{\gamma}_t^E = \max\{\bar{\gamma},\, r^L(\bar{\gamma})\}$. It will never offer more than that, since this proposal makes L indifferent between either accepting it or rejecting it and getting the default payoff. That is, L could not be strictly better off by rejection. Hence, $d_t^L = 1$. Moreover, V anticipates this and expects E will use discretion at the implementation, so that the actual spending moves closer to γ^u. Concretely, E sets

$$\begin{aligned} \gamma_t^s &= \min\left\{\gamma_t^u,\ \tilde{\gamma}_t^s + (1-\delta)\left[\tau - \tilde{\gamma}_t^s\right]\right\}, \\ &= \min\left\{\gamma_t^u,\ \tau - \delta[\tau - \tilde{\gamma}_t^s)]\right\}. \end{aligned}$$

Rationality of expectations implies the optimal solution of E coincides with V's expected equilibrium policy. Finally, notice that $\tilde{\gamma}_t^E$ will be lower, the closer $\bar{\gamma}$ is to γ^*. In effect, $\frac{\partial \tilde{\gamma}_t^E}{\partial \bar{\gamma}} \geq 0$ for all $\bar{\gamma} \geq \gamma^*$ and $\frac{\partial \tilde{\gamma}_t^E}{\partial \bar{\gamma}} < 0$ for $\bar{\gamma} < \gamma^*$.

In words, Proposition 2 says that separation of powers moderates electoral cycles for intermediate reversion levels (i.e., for $\bar{\gamma} \in (r^L(\gamma^u),\, \gamma^u)$), but not for extreme levels, where cycles are just like under unified government.

[16]Single-peakedness follows from the strict concavity of $E_t[v(\gamma_t,\, \theta_t)]$ and $\mu_{t+1}^E(\gamma_t)$.

Furthermore, Proposition 2 shows that the existence of discretion at the implementation reduces the moderating influence of the legislature. In effect, looking at (5.15), it is clear that γ_t^s approaches γ_t^u as δ falls. In the limit, when all tax resources can be arbitrarily reassigned, i.e. when $\delta = 0$, $\gamma_t^s = \gamma_t^u$. On the other hand, if $\delta = 1$, then $\gamma_t^s = \tilde{\gamma}_t^s$. This discussion is formally summarized in Corollary 2, whose proof can be immediately derived from Proposition 2.

Finally, Figure 5.2 below shows the shape of γ_t^s as a function of δ:

Figure 5.2: Closed rule.

For a status quo $\bar{\gamma} \in (r^L(\gamma^u), \gamma^u)$, Figure 5.2a shows that γ_t^s coincides with E's ideal policy for $\delta \leq \delta^{crit}(\bar{\gamma})$. However, for $\delta > \delta^{crit}(\bar{\gamma})$, γ_t^s decreases monotonically as δ rises. γ_t^s reaches $\tilde{\gamma}_t^s$ when $\delta = 1$, but even in this case it stays above γ^* (except of course for the non-generic case in which $\bar{\gamma} = \gamma^*$). For $\bar{\gamma} \in [0, r^L(\gamma^u)] \cup [\gamma^u, \tau]$, Figure 5.2b shows that γ_t^s is completely insensitive to the value of δ.

(ii) Open rule

Suppose now the legislature can introduce any amendment into the executive's proposal, but the executive has veto power over it. Under this institutional structure, the role of each incumbent is in fact reversed. That is, L becomes the actual agenda-setter, while E reduces to a veto player. The main result is the following:

Proposition 3 *Suppose there is separation of powers. If the legislature can introduce any amendment into the executive's proposal, then there exists a unique pure strategy equilibrium* $\hat{\lambda}^s$ *in* G *such that* $d_{t+1}^{L,s} = d_t^{L,s} = 1$, $\tilde{\gamma}_{t+1}^{E,s} = \gamma_{t+1}^s = \gamma^*$,

$$\tilde{\gamma}_t^{E,s} = \begin{cases} \min\left\{\bar{\gamma},\ r^E(\bar{\gamma})\right\} & \text{if } \bar{\gamma} \in (\widehat{\gamma},\ r^E(\widehat{\gamma})), \\ \widehat{\gamma} & \text{otherwise,} \end{cases} \tag{5.17}$$

$$\gamma_t^s = \min\left\{\gamma_t^u,\ \tau - \delta[\tau - \tilde{\gamma}_t^s]\right\}, \tag{5.18}$$

and $\lambda^V = 1$ *if and only if* $\theta_t^e = g_t/\gamma_t^u \geq \bar{\theta}$, *where* $\widehat{\gamma} = \max\left\{0,\ \frac{\gamma^* - (1-\delta)\tau}{\delta}\right\}$.

Corollary 3 *Electoral cycles depend on the status quo and overview power of legislature,*

1. *If* $\delta > 1 - \alpha$ *and* $\bar{\gamma} \in \left[0, \, \widehat{\gamma}\right] \cup \left[r^E(\widehat{\gamma}), \, \tau\right]$, *then* $\Delta^s(\bar{\gamma}, \, \delta) = \Delta^*$*;*

2. *If either* $\delta > 1 - \alpha$ *and* $\bar{\gamma} \in (\widehat{\gamma}, \, r^E(\widehat{\gamma}))$ *or* $\delta \leq 1 - \alpha$, *then* $\Delta^* < \Delta^s(\bar{\gamma}, \, \delta) \leq \Delta^u$*;*

3. *Given* $\bar{\gamma} \in [0, \, \tau]$, $\Delta^s(\bar{\gamma}, \, \delta)$ *is non-increasing in* δ.

To derive Proposition 3, the analysis is similar to (i). The equilibrium at the post-electoral period $t+1$ and the optimal response of V to the observation of g_t are exactly the same.

With respect to the bargaining process carried out at period t, the only difference is who has the effective power to make final offers. Here the actual agenda-setter is the legislative leader, instead of the executive incumbent. It will be clear below that this reduces considerably the electoral distortion on γ_t, compared with the case studied in the previous subsection, since it curtails E's power over the budget composition.

Given the beliefs of V over the competence of E, for any level of authorized expenditures $\tilde{\gamma}_t$, the policy implemented will be $\gamma_t = \min \left\{\gamma_t^u, \, \tau - \delta[\tau - \tilde{\gamma}_t]\right\}$. That is, E will set γ_t at its most-preferred policy or, if this were not possible, it will use at the implementation the maximum degree of discretion to achieve an alternative as close as possible to γ^u.

For a given value of δ, let $\widehat{\gamma}$ be implicitly defined by the following condition: $\tau - \delta[\tau - \widehat{\gamma}] = \gamma^*$; or set it equal to zero if $\gamma^* \leq (1-\delta)\,\tau$. That is, let $\widehat{\gamma} = \max\left\{0, \, \frac{\gamma^* - (1-\delta)\tau}{\delta}\right\}$. It is clear that $\widehat{\gamma} > 0$ if and only if $\delta > 1 - \alpha$. Then, for $\delta > 1-\alpha$ and $\bar{\gamma} \in [0, \, \widehat{\gamma}] \cup [r^E(\widehat{\gamma}), \, \tau]$, the legislature's leader would amend any executive's proposal $\tilde{\gamma}_t^E \neq \widehat{\gamma}$, by setting $\tilde{\gamma}_t^L = \widehat{\gamma}$. This amendment satisfies the incentive constraint $\tilde{\pi}^E(\tilde{\gamma}_t^L) \geq \tilde{\pi}^E(\bar{\gamma})$ (see Figure 5.1). Therefore, it cannot be vetoed by E. Understanding this, E weakly prefers to make such an offer rather than to propose a different spending and lose approval in the legislature. L also prefers to accept it rather than to reject it, because by definition $\widehat{\gamma}$ ensures its ideal policy is realized.

A similar reasoning can be made if either $\delta > 1-\alpha$ and $\bar{\gamma} \in (\widehat{\gamma}, \, r^E(\widehat{\gamma}))$ or $\delta \leq 1-\alpha$. However, in these cases $\widehat{\gamma}$ does not satisfy the incentive constraint of E. That is, $\tilde{\pi}^E(\widehat{\gamma}) < \tilde{\pi}^E(\bar{\gamma})$. Therefore, L cannot achieve its ideal policy γ^*. For any level of authorized expenditures $\tilde{\gamma}_t^s$, it follows that $\gamma_t^s > \gamma^*$. Nevertheless, following the logic of the agenda setter, L restricts player E to its reservation utility, by amending any proposal $\tilde{\gamma}_t^E \neq \min\left\{\bar{\gamma}, \, r^E(\bar{\gamma})\right\}$. Hence, this policy is proposed in equilibrium and, therefore, $\Delta^* < \Delta^s(\bar{\gamma}, \, \delta) \leq \Delta^u$. Furthermore, $\Delta^s(\bar{\gamma}, \, \delta) = \Delta^u$ only if $\bar{\gamma} = \gamma^u$ or $\delta \leq \delta^{crit}(\bar{\gamma})$.

In words, Proposition 3 says that, when there exists agenda-setting authority and open rule, separation of powers completely eliminates the electoral cycles on γ_t for low and high reversion levels. Contrary, for intermediate values of $\bar{\gamma}$ and for δ sufficiently greater than zero, the electoral cycle in public consumption expenditures cannot be eliminated, but its magnitude is reduced. As in the previous section, this moderating force decreases when the executive enjoys discretion at the implementation stage (see Figure 5.3).

Figure 5.3: Open rule.

As a final remark, notice that for low and high values of $\bar{\gamma}$ the results with and without amendments are exactly the opposite. While the former provides consumption expenditures close to the first best allocation, the second supplies the same predictions as unified government. The explanation for this is based on who is the actual veto player (alternatively, the agenda-setting authority) in each case, and by the fact that the veto player has greatest power when the reversion policy is very near its preferred policy.

For the results derived in Proposition 1, for instance, one can draw a graph showing that $\tilde{\gamma}_t^s$ starts at γ^u, for $\bar{\gamma} = 0$, then it eventually starts falling, reaching γ^* as $\bar{\gamma}$ approaches γ^*, and then it starts rising again to γ^u. Contrary, for Proposition 2, the graph has the inverse shape, starting at $\widehat{\gamma}$, then rising towards γ^u, and reaching it when $\bar{\gamma} = \gamma^u$, before starting to fall again. (See Figure 5.4.)

Figure 5.4: Authorized expenditures.

This behavior of $\tilde{\gamma}_t^s$ explains the opposite results obtained under the closed and the open rule.

5.4 Final remarks

This simple moral-hazard model of PBC (à la Lohmann) predicts optimal equilibrium policy during off-electoral periods, but not just ahead of the elections. Policy distortions over the composition of government spending occur just before elections because the incumbent's incentive to appear competent during these periods induces overspending in the public consumption good (the more visible good), while simultaneously reducing below the socially optimal level the spending on the public capital good.

The fact that the executive incumbent is unable to credible compromise to the optimal allocation policy is at the heart of electoral distortions. It turns out that this problem is in fact generated by concentration of powers, which allows the executive to choose any policy it desires. Instead, when there exists more than one policy-maker and appropriate checks and balances, separation of powers, by requiring joint agreement in decision-making, restricts its capacity of unilateral moves and, therefore, works as a commitment device that reduces the size of electoral distortions, making all players better off (included the executive incumbent).[17]

On the other hand, this model can be seen as a first step to understand how the incumbent chooses among different fiscal policy instruments or, alternatively, why it uses some of them more frequently in some countries than in others.[18] Even though the fiscal policy includes several items, like taxes, expenditure composition and debt, there is no general model of rational PBC that explains how politicians choose among them. Following the logic of this model, the main prediction on this point is that institutional details play an important role in the selection. Concretely, one would expect that the executive incumbent uses those policy instruments where it has greater agenda-setting authority. Nevertheless, it is left for a future research to formally explore this conjecture, as well as its empirical validity.

[17]Separation of powers should affect adverse selection models of PBC (à la Rogoff) in a similar way. The legislature basically tries to avoid distortions in the allocation of budget resources. This should reduce the electoral distortions of fiscal policy, preserving the signaling role of the provision of public goods.

[18]For instance, tax cuts before elections seems to be more frequent in OECD countries, while changes of the expenditure composition and budget deficits are usually observed in Latin American countries. For more on that, see for example Block (2002), Persson and Tabellini (2002) and Shi and Svensson (2002a, 2002b).

5.5 Appendix

Proof of Proposition 1: To derive Proposition 1, notice first that the sequence of two-period games postulated in section 2 is well-defined, in the sense that each individual game is uncorrelated with any other member of the sequence. To see that, consider V's expected utility at, for example, post-electoral period $t+3$. By voting rule (5.6), this determines V's vote at $t+2$. However, since competence follows a MA(1) process, V's expected utility at $t+3$ is not affected by E's competence at $t+1$: $E_{t+1}[\theta_{t+3}|\ \theta_{t+1}] = E_{t+1}[\theta_{t+3}] = \bar{\theta}$. Therefore, period $t+1$ in G is independent of the continuation game. This implies I has no incentives to manipulate V's perception of its competence at $t+1$. Consequently, actual expenditures on g_{t+1} are $\gamma^u_{t+1} = \gamma^*_{t+1}$.

Consider now electoral period t. Given λ^V in (5.6) and γ^u_{t+1}, V votes for I if and only if $E_t[\theta_{t+1}|\ I] \geq E_t[\theta_{t+1}|\ \hat{E}(t)]$. Letting $\theta^e_t \equiv E_t[\theta_t|\ I]$ denote voters' expectations, and noting that the only information on potential replacement is average competency, $\lambda^V_t = 1$ if and only if $\theta^e_t \geq \bar{\theta}$.

Since at election time V knows g_t, but it does not observe ε_t, it has to estimate θ^e_t. Let γ^e_t be the solution, expected by V, of the incumbent's optimization problem at date t.[19] Using equation (5.3) and Bayes rule, V estimates I's competence by

$$\begin{aligned} \theta^e_t &= E_t[\theta_t \mid g_t] = \int_{\theta_t \in \Theta_t} \theta_t\ \tilde{F}(\theta_t \mid g_t)\ d\theta_t, \\ &= \frac{g_t}{\gamma^e_t}\ \underbrace{\int_{\theta_t \in \Theta_t} \tilde{F}(\theta_t \mid g_t)\ d\theta_t}_{=1} = \frac{g_t}{\gamma^e_t}. \end{aligned} \tag{5.19}$$

The probability I attaches to being in office in period $t+1$, μ^I_{t+1}, is as follows. By (5.3) and (5.19), $\theta^e_t \geq \bar{\theta}$ if and only if $\theta_t \geq \frac{\bar{\theta}\gamma^e_t}{\gamma_t}$. Using (5.10),

$$\mu^I_{t+1}(\gamma_t) = 1 - \tilde{F}\left(\frac{\bar{\theta}\,\gamma^e_t}{\gamma_t}\right). \tag{5.20}$$

Thus, I's maximization problem at period t can be written as,

$$\max_{\gamma_t}\ \ E_t\left\{\theta_t(\gamma_t)^\alpha\,(\tau-\gamma_t)^{1-\alpha} + \beta\,\mu^I_{t+1}\,\chi\right\}, \tag{5.21}$$

subject to (5.20). Taken the first order condition with respect to γ_t, we have

$$\left(\frac{\gamma_t}{\tau-\gamma_t}\right)^\alpha \left[1-\alpha\left(\frac{\tau-\gamma_t}{\gamma_t}+1\right)\right] = \frac{\beta\,\chi\,\tilde{F}'\gamma^e_t}{(\gamma_t)^2}. \tag{5.22}$$

[19] Since I does not observe its competence before choosing the expenditure composition, γ^e_t cannot depend on θ_t.

In equilibrium, $\gamma_t = \gamma_t^e$, since actual and expected decisions coincide. Denote equilibrium $\gamma_t \equiv \gamma_t^u$. Therefore, (5.22) can be re-written as

$$\left(\frac{\gamma_t^u}{\kappa_t^u}\right)^{\alpha} (\gamma_t^u - \alpha\tau) = \beta\, \chi\, \tilde{F}'. \tag{5.23}$$

Notice that the right hand side in (5.23) is positive. Thus, $(\gamma_t^u - \alpha\tau) > 0$, which means $\gamma_t^u > \alpha\tau = \gamma_t^*$ and $\kappa_t^u < (1-\alpha)\tau = \kappa_t^*$. Further, in equilibrium, $\mu_{t+1}^I = 1 - \tilde{F}\left(\bar{\theta}\right) = \frac{1}{2}$. Finally, uniqueness of γ_t^u follows from the strictly concavity of both (5.20) and (5.21).

Proof of Corollary 1: The first part is immediately derived from Proposition 1. As to the second part, to see that Δ^u is strictly increasing in χ, notice first that $\partial\Delta^u/\partial\chi = \partial\gamma_t^u/\partial\chi$. Therefore, totally differentiating the first order condition in (5.23) with respect to χ, it follows that,

$$\frac{\partial\gamma_t^u}{\partial\chi} = \frac{\beta\tilde{F}'}{\left(\frac{\gamma_t^u}{\tau-\gamma_t^u}\right)^{\alpha}\left[\frac{\alpha\tau(\gamma_t^u-\alpha\tau)}{\gamma_t^u(\tau-\gamma_t^u)}+1\right]},$$

which is strictly greater than zero. Following the same reasoning, it can be shown that,

$$\frac{\partial\gamma_t^u}{\partial\beta} = \frac{\chi\tilde{F}'}{\left(\frac{\gamma_t^u}{\tau-\gamma_t^u}\right)^{\alpha}\left[\frac{\alpha\tau(\gamma_t^u-\alpha\tau)}{\gamma_t^u(\tau-\gamma_t^u)}+1\right]},$$

which is also positive.

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