1 Cash in advance model

Cash in advance

- Adding money to model
  - Somewhat ad hoc method
  - NOT micro foundations for why people hold money
  - we assume that they must

- Assume that one needs money to purchase consumption good
- Carry money over from pervious period (plus some possible transfers)
- velocity is constant (one cycle per period)
- Story
- I show two ways to solve the models

Model of Cooley and Hansen

- Unit mass of identical agents
- Will assume indivisible labor (not a big deal)
- Agents will need money to make consumption purchases
  - money held over from previous period
- Government can make direct lump-sum transfers or taxes of money
- Addition of money means that second welfare theorem need not hold
  - individual decisions based on
    * aggregate amount of money
    * price level

The model

- Households' maximize
  \[ E_0 \sum_{t=0}^{\infty} \beta^t u(c^i_t, h^i_t) \]

- where
  \[ u(c^i_t, h^i_t) = \ln c^i_t + \left[ A \frac{\ln(1 - h_0)}{h_0} \right] h^i_t \]
• Production takes place with production function
  \[ y_t = \lambda_t K_t^\theta H_t^{1-\theta} \]

• Technology follows
  \[ \ln(\lambda_{t+1}) = \gamma \ln(\lambda_t) + \varepsilon_{t+1} \]

The model

• Competitive factor markets imply that
  \[ w_t = (1 - \theta) \lambda_t K_t^\theta H_t^{1-\theta}, \]
  and
  \[ r_t = \theta \lambda_t K_t^{\theta-1} H_t^{1-\theta}. \]

• Aggregation conditions are
  \[ H_t = \int_0^1 h_i^t di, \]
  and
  \[ K_t = \int_0^1 k_i^t di. \]

The model

• The households budget constraint is
  \[ c_t^i + k_{t+1}^i + \frac{m_t^i}{p_t} = w_t h_t^i + r_t k_t^i + (1 - \delta) k_{t-1}^i + \frac{m_{t-1}^i + (g_t - 1) M_{t-1}}{p_t} \]
  – \( \frac{m_t^i}{p_t} \) is the real value of money carried into the next period
  – \( m_{t-1}^i + (g_t - 1) M_{t-1} \) is money from the previous period plus transfers (or taxes) from the government
  – \( g_t \) is the gross growth rate of money: \( M_t = g_t M_{t-1} \)

• The cash-in-advance constraint is
  \[ p_t c_t^i \leq m_{t-1}^i + (g_t - 1) M_{t-1} \]
  – This is an additional constraint on consumption
  – Want it to always hold (with equality)
  – Need to have gross money growth greater than discount factor, \( \beta \)

Normalization issues

• Models are valid close to stationary states
Need to have stable models so that they stay near SS

Money is a problem
- Money is a stock
- An increase in the growth rate imply change of level
- No reason to return to old level

Two methods of normalizing
- Measure all nominal variables relative to the aggregate money stock
- Measure all nominal variables in real terms (divided by price level)

Normalization issues

Method of Cooley-Hansen
- They divide all nominal variables by money stock
- Define \( \hat{p}_t = p_t/M_t \), \( \hat{m}_t^i = m_t^i/M_t \), and \( M_t/M_t = 1 \)
- Cash-in-advance constraint is
  \[
  \frac{p_t}{M_t} c_t^i = \frac{m_{t-1}^i + (g_t - 1) M_{t-1}}{M_t}
  \]
  or
  \[
  \hat{p}_t c_t^i = \frac{m_{t-1}^i + (g_t - 1) M_{t-1}}{g_t M_{t-1}}
  \]
  \[
  \hat{p}_t c_t^i = \frac{\hat{m}_{t-1}^i + (g_t - 1)}{g_t}
  \]
- Budget constraint is
  \[
  c_t^i + k_{t+1}^i + \frac{\hat{m}_t^i}{\hat{p}_t} = w_t h_t^i + r_t k^i \frac{1 - \delta}{g_t} + \frac{\hat{m}_{t-1}^i + (g_t - 1)}{g_t \hat{p}_t}
  \]

Normalization issues

Real balance method
- Divide all nominal variables by the price level
- They usually show up in real equations in this form anyway
- A family’s real balances are
  \[
  \frac{m}{p} = \frac{m_t^i}{p_t},
  \]
  and the economy real balances are
  \[
  \frac{M}{p} = \frac{M_t}{p_t}.
  \]
Some care needs to be taken with the lagged money variables. In a stationary state, $\bar{g} = \pi$. In the stationary state

\[
\frac{m_{t-1}^i}{p_t} = \frac{m_{t-1}^i}{\pi_{t-1}} = \frac{m/p}{\pi} = \frac{m/p}{\bar{g}}.
\]

Normalization issues

- Cash-in-advance constraint is

\[
c_t^i = \frac{m_{t-1}^i}{P_{t-1}} P_{t-1} + (g_t - 1) \frac{M_{t-1}}{P_t} \frac{P_{t-1}}{P_t} P_{t-1}
\]

\[
c_t^i = \frac{m_{t-1}^i}{P_{t-1}} \frac{1}{\pi_t} + (g_t - 1) \frac{M_{t-1}}{P_t} \frac{1}{\pi_t}
\]

- The flow budget constraint (after removing the cash in advance constraint) is

\[
k_{t+1}^i + \frac{m_i^i}{P_t} = w_t h_t^i + r_t k_t^i + (1 - \delta) k_t^i
\]

- Will have variables $P_t$ and $M_t$ that could have a unit root
  - not a real problem because they always appear together
    * and they are co-integrated
  - real variables of model do not have unit roots

- Go back to Cooley-Hansen’s way

Full model

- Households max

\[
\max \sum_{t=0}^{\infty} \left( \beta^t \ln c_t^i + \left[ A \ln(1 - h_0) \right] h_t^i \right)
\]

subject to the budget constraints

\[
\frac{c_{t+1}}{p_{t+1}} = \frac{\tilde{m}_{t-1}^i + (g_t - 1)}{g_t \bar{p}_t}
\]

\[
c_t^i + k_{t+1}^i + \frac{\tilde{m}_t^i}{\bar{p}_t} = ((1 - \theta) \lambda_t^{-1} K_{t+1}^{-\theta}) h_t^i + \left( \theta \lambda_t^{\theta-1} H_{t+1}^{1-\theta} \right) k_t^i
\]

\[
+ (1 - \delta) k_t^i + \frac{\tilde{m}_{t-1}^i + (g_t - 1)}{g_t \bar{p}_t}
\]
• The law of motion for the stochastic shock
  \[ \ln \lambda_{t+1} = \gamma \ln \lambda_t + \varepsilon_{t+1} \]

Full model
• The growth rate for money that is either a stationary state rule,
  \[ g_t = \bar{g} \]
  or a stochastic rule
  \[ \ln g_{t+1} = (1 - \pi) \ln \bar{g} + \pi \ln g_t + \varepsilon_{t+1} \]

• The aggregation conditions for an equilibrium are
  \[ K_t = k_t^i \]
  \[ H_t = h_t^i \]
  \[ C_t = c_t^i \]

  and
  \[ \hat{M}_t = \hat{m}_t^i = 1 \]

Solving the model
• The first order conditions and constraints for households
  \[ \frac{1}{\beta} = E_t \frac{w_t}{w_{t+1}} [(1 - \delta) + r_{t+1}] \]
  \[ \frac{B \bar{g}}{w_t \hat{p}_t} = -\beta E_t \frac{1}{\hat{p}_{t+1} c_{t+1}^i} \]
  \[ \hat{p}_t c_t^i = \frac{\hat{m}_{t-1}^i + g_t - 1}{g_t} \]
  \[ k_{t+1}^i + \frac{\hat{m}_t^i}{\hat{p}_t} = (1 - \delta) k_t^i + w_t h_t^i + r_t k_t^i \]

• Factor market conditions
  \[ w_t = (1 - \theta) \lambda_t \left[ \frac{K_t}{H_t} \right]^{\theta} \]
  and
  \[ r_t = \theta \lambda_t \left[ \frac{K_t}{H_t} \right]^{\theta-1} \]

• Equilibrium conditions
  \[ C_t = c_t^i, \quad H_t = h_t^i, \quad K_{t+1} = k_{t+1}^i, \text{ and } \hat{M}_t = \hat{m}_t^i = 1 \]

Stationary state
The equations for finding the stationary state

\[
\begin{align*}
\frac{1}{B} &= (1 - \delta) \times r \\
\frac{\beta}{\overline{w}} &= -\frac{\beta}{\gamma C} \\
\hat{\rho} \overline{C} &= 1 \\
\frac{1}{\hat{\rho}} &= (\tau - \delta) \overline{K} + \overline{w} \overline{H} \\
\overline{w} &= (1 - \theta) \left( \frac{K}{H} \right)^\theta \\
\tau &= \theta \left( \frac{K}{H} \right)^{\theta - 1}
\end{align*}
\]

Stationary state

- Solving the equations of the stationary state give

\[
\begin{align*}
\tau &= \frac{1}{\beta} - (1 - \delta) \\
\overline{w} &= (1 - \theta) \left[ \frac{K}{H} \right]^\theta = (1 - \theta) \left[ \frac{\tau}{\theta} \right]^{\frac{\theta}{\theta - 1}} \\
\overline{C} &= -\frac{\beta \overline{w}}{\gamma B} \\
\hat{\rho} &= \frac{1}{\overline{C}} \\
\overline{K} &= \frac{\overline{C}}{\gamma - \delta} \\
\overline{H} &= \left( \frac{\tau}{\theta} \right)^{-\frac{\theta}{\theta - 1}} K \\
\overline{V} &= \overline{C} + \delta \overline{K}
\end{align*}
\]

Stationary state

- Parameter values are \( \beta = .99, \delta = .025, \theta = .36, A = 1.72, \) and \( h_0 = .583, \) so \( B = -2.5805 \)

- These give stationary state values of
Notice how the growth rate of money affects real variables

It is also possible to calculate the welfare loss from inflation:

\[
utility = \ln \left( \frac{0.9095}{g} \right)^2 - \frac{2.5805 \times 0.3302}{g} - 9.486
\]

Solving the dynamic model: version 1

Cooley and Hansen used linear quadratic method

Problem: there are two economy wide variables, \( K_t \) and \( \hat{p}_t \)

These do not come directly from individual maximization problems

Come from aggregation or equilibrium conditions

Individual maximization problems do depend on these

(we will also want to remove labor (both individual and aggregate) from model)

Solving the dynamic model: version 1

How to proceed

elinate consumption from optimization problem using c-i-a constraint

\[
\max E_0 \sum_{t=0}^{\infty} \left( \beta^t \ln \left[ \frac{\hat{m}_t + (g_t - 1)}{g_t \hat{p}_t} \right] + \left[ \frac{\ln(1 - h_0)}{h_0} \right] k_t \right)
\]

Using remaining budget constraint

\[
k_{t+1} + \frac{\hat{m}_t}{\hat{p}_t} = \left( (1 - \theta) \lambda_t K_t^\theta H_t^{-\theta} \right) h_t + \left( \theta \lambda_t K_t^{\theta - 1} H_t^{1-\theta} \right) k_t + (1 - \delta) k_t
\]

and simplify to get

\[
k_{t+1} - (1 - \delta) k_t + \frac{\hat{m}_t}{\hat{p}_t} = \left( \lambda_t K_t^\theta H_t^{1-\theta} \right) \left[ (1 - \theta) h_t + \theta k_t \right]
\]

Solving the dynamic model: version 1
• Sum across households to get

\[ K_{t+1} + \frac{1}{p_t} = \lambda_t K_t^\theta H_{t}^{1-\theta} + (1 - \delta) K_t \]

which can be solved for aggregate labor as

\[ H_t = \left[ \frac{K_{t+1} - (1 - \delta) K_t + \frac{1}{p_t}}{\lambda_t K_t^\theta} \right] \]

• Individual labor is then

\[ h_t = \frac{k_{t+1}^i - (1 - \delta) k_t^i + \frac{\hat{m}_{t+1}^i}{p_t} - \theta \left[ K_{t+1} - (1 - \delta) K_t + \frac{1}{p_t} \right] \frac{k_t^i}{K_t}}{(1 - \theta) \left[ K_{t+1} - (1 - \delta) K_t + \frac{1}{p_t} \right]^{1-\theta} \left[ \lambda_t K_t^\theta \right]^{1-\theta}} \]

Solving the dynamic model: version 1

• Put all this into the objective function

\[
\max_{k_t^i, \hat{m}_t^i} \sum_{t=0}^\infty \left( \beta^t \ln \left[ \frac{\hat{m}_{t-1}^i + (g_t - 1)}{g_t p_t} \right] + \left[ A \ln(1 - h_0) \right] \times \left[ k_{t+1}^i - (1 - \delta) k_t^i + \frac{\hat{m}_{t+1}^i}{p_t} - \theta \left[ K_{t+1} - (1 - \delta) K_t + \frac{1}{p_t} \right] \frac{k_t^i}{K_t} \right] \right)
\]

subject to the budget constraints

\[ k_{t+1}^i = k_t^i \]
\[ \hat{m}_{t+1}^i = \hat{m}_t^i \]
\[ \ln(\lambda_{t+1}) = \gamma \ln(\lambda_t) + \epsilon_{t+1}^\lambda \]
\[ \ln g_{t+1} = (1 - \pi) \bar{g} + \pi \ln g_t + \epsilon_{t+1}^g \]

Solving the dynamic model: version 1

• State variables: \( x_t^i = [ 1 \quad \lambda_t \quad k_t^i \quad \hat{m}_{t-1}^i \quad g_t \quad K_t ] \)

• Control variables: \( y_t^i = [ k_{t+1}^i \quad \hat{m}_t^i ] \)

• Economy wide variables \( Z_t = [ K_{t+1} \quad \hat{p}_t ] \)

• Write the linear quadratic objective function as

\[
\begin{bmatrix} x_t^i & y_t^i & Z_t \end{bmatrix} Q \begin{bmatrix} x_t \\ y_t \\ Z_t \end{bmatrix}
\]
- Given this objective function, we want to solve a Bellmans equation of the form

\[ x_t' P x_t = \max_{y_t} \begin{bmatrix} x_t' \\ y_t' \\ Z_t' \end{bmatrix} Q \begin{bmatrix} x_t \\ y_t \\ Z_t \end{bmatrix} + \beta E_0 \begin{bmatrix} x_{t+1}' P x_{t+1} \end{bmatrix} \]

subject to the budget constraints

\[ x_{t+1} = A x_t + B y_t + C Z_t + D \varepsilon_{t+1} \]

Solving the dynamic model: version 1

- rewrite the matrix \( Q \) as

\[ Q = \begin{bmatrix} R & W' & X' \\ W & T & N' \\ X & N & S \end{bmatrix}, \]

- write

\[ \begin{bmatrix} x_t' \\ y_t' \\ Z_t' \end{bmatrix} Q \begin{bmatrix} x_t \\ y_t \\ Z_t \end{bmatrix} \]

as

\[ x_t' R x_t + y_t' T y_t + Z_t' S Z_t + 2 y_t' W x_t + 2 Z_t' X x_t + 2 Z_t' N y_t. \]

- The last part of the Bellmans equation can be written as

\[ \beta E_0 \begin{bmatrix} x_{t+1}' P x_{t+1} \end{bmatrix} = \beta E_0 \left( (A x_t + B y_t + C Z_t + D \varepsilon_{t+1})' P (A x_t + B y_t + C Z_t + D \varepsilon_{t+1}) \right) \]

Solving the dynamic model: version 1

- First order condition are

\[ 0 = T y_t + W x_t + N' Z_t + \beta \left[ B' P A x_t + B' P B y_t + B' P C Z_t \right] \]

or

\[ (T + \beta B' P B) y_t = - (W + \beta B' P A) x_t - (N + \beta B' P C) Z_t \]

- When \( (T + \beta B' P B) \) is invertible, the linear policy function is

\[ y_t = - (T + \beta B' P B)^{-1} (W + \beta B' P A) x_t - (T + \beta B' P B)^{-1} (N + \beta B' P C) Z_t \]
which we can write as

\[ y_t = F_1 x_t + F_2 Z_t, \]

with

\[
\begin{align*}
F_1 &= -(T + \beta B'PB)^{-1}(W + \beta B'PA) \\
F_2 &= -(T + \beta B'PB)^{-1}(N + \beta B'PC)
\end{align*}
\]

Solving the dynamic model: version 1

- The value function \( P \) that we want fulfills

\[
\begin{align*}
x_t'Px_t &= x_t' \begin{bmatrix} F_1 x_t + F_2 Z_t \\ Z_t \end{bmatrix} Q \begin{bmatrix} F_1 x_t + F_2 Z_t \\ Z_t \end{bmatrix} + \beta \left[ (A + BF_1)x_t + (BF_2 + C)Z_t \right]' P \left[ (A + BF_1)x_t + (BF_2 + C)Z_t \right]
\end{align*}
\]

- Unfortunately, the \( Z_t \) variables are still a problem

Solving the dynamic model: version 1

- Handling the economy wide variables

- We can aggregate (integrate over) the controls to get

\[
\int_0^1 y_t' di = \begin{bmatrix} \int_0^1 k^t_{i+1} di \\ \int_0^1 \hat{m}^t_{i+1} di \end{bmatrix} = \begin{bmatrix} K^t_{i+1} \\ 1 \end{bmatrix}
\]

- An aggregated version of the policy function is

\[
\int_0^1 y_t' di = F_1 \int_0^1 x_t' di + F_2 Z_t
\]

or

\[
\begin{bmatrix} K^t_{i+1} \\ 1 \end{bmatrix} = F_1 \int_0^1 x^t_{i+1} di + F_2 Z_t
\]

Solving the dynamic model: version 1

- Since

\[
x^t_i = \begin{bmatrix} 1 & \lambda_t & k^t_i & \hat{m}^t_{i-1} & g_t & K_t \end{bmatrix} ',
\]

- The integral of this vector is

\[
\tilde{x}_t = \int_0^1 x^t_{i+1} di = \begin{bmatrix} 1 & \lambda_t & K_t & 1 & g_t & K_t \end{bmatrix},
\]
we can construct a matrix

\[ G = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 
\end{bmatrix}, \]

So that \( \hat{x}_i = Gx_i \), for all \( i \)

Solving the dynamic model: version 1

The aggregate version of the policy function is

\[
\begin{bmatrix}
K_{t+1} \\
\hat{p}_t
\end{bmatrix} = F_1 G x_i + F_2 \begin{bmatrix}
K_{t+1} \\
\hat{p}_t
\end{bmatrix}
\]

This equation can be solved for the vector \( \begin{bmatrix} K_{t+1} \\ \hat{p}_t \end{bmatrix} \) as

\[
\begin{bmatrix}
K_{t+1} \\
\hat{p}_t
\end{bmatrix} = F_2^{-1} \begin{bmatrix}
K_{t+1} \\
\hat{p}_t
\end{bmatrix} - F_2^{-1} F_1 G x_i
\]

or as

\[
\begin{bmatrix}
K_{t+1} \\
\hat{p}_t
\end{bmatrix} = J \begin{bmatrix}
K_{t+1} \\
\hat{p}_t
\end{bmatrix} + H x_i,
\]

Solving the dynamic model: version 1

Recalling that the first element of \( x_i \) is always 1, one can find a function of the form

\[ Z_t = F_3 x_i \]

Here

\[
F_3 = \begin{bmatrix}
\frac{H_{11} + J_{12}}{1-J_{11}} & \frac{H_{12}}{1-J_{11}} \\
H_{21} + J_{22} + \frac{J_{21}(H_{11} + J_{12})}{1-J_{11}} & H_{22} + \frac{J_{21}H_{12}}{1-J_{11}} \\
\frac{H_{13}}{1-J_{11}} & H_{23} + \frac{J_{21}H_{13}}{1-J_{11}} \\
\frac{H_{14}}{1-J_{11}} & H_{24} + \frac{J_{21}H_{14}}{1-J_{11}} \\
\frac{H_{15}}{1-J_{11}} & H_{25} + \frac{J_{21}H_{15}}{1-J_{11}} \\
\frac{H_{16}}{1-J_{11}} & H_{26} + \frac{J_{21}H_{16}}{1-J_{11}}
\end{bmatrix}.
\]

Solving the dynamic model: version 1
Bellman equation is

\[
P = \begin{bmatrix} I_x & F_1' + F_2'F_2 & F_3' \end{bmatrix} Q \begin{bmatrix} I_x \\ F_1 + F_2F_3 \\ F_3 \end{bmatrix} + \beta [(A + BF_1) + (BF_2 + C) F_3]' P [(A + BF_1) + (BF_2 + C) F_3]
\]

- To solve, choose \( P_0 \)
- Find \( F_1^0, F_2^0 \) and using these find \( F_3^0 \)
- Use these along with \( P_0 \) in the above equation to find \( P_1 \)
- Repeat until conversion is close enough

Alternative method for solving

- Log-linearization of the model
  - First order conditions
  - budget constraints
  - market equilibrium conditions (competitive or not)
  - Aggregation and other equilibrium conditions

- Economy wide variables are not, in general, a problem
  - optimization already done
  - model usually in aggregate variables

Cash in advance Model

- First order conditions

\[
\frac{1}{\beta} = E_t \frac{w_t}{w_{t+1}} [(1 - \delta) + r_{t+1}],
\]

\[
\frac{B\gamma}{w_{t+1}p_{t}} = -\beta E_t \frac{1}{\hat{p}_{t+1}c_{t+1}^i},
\]

- the cash in advance constraint

\[
\hat{p}_t c_t^i = \frac{\hat{m}^i_{t-1} + g_t - 1}{g_t},
\]

- the flow budget constraint

\[
k_{i+1}^i + \frac{\hat{m}^i_t}{\hat{p}_t} = (1 - \delta) k_t^i + w_t h_t^i + r_t k_t^i.
\]
Cash in advance Model

- Factor market conditions
  \[ w_t = (1 - \theta) \lambda_t \left( \frac{K_t}{H_t} \right)^\theta, \]
  and
  \[ r_t = \theta \lambda_t \left( \frac{K_t}{H_t} \right)^{\theta - 1}. \]

- Equilibrium and aggregation conditions are
  \[ C_t = c^i_t, \quad H_t = h^i_t, \]
  \[ K_{t+1} = k^i_{t+1}, \quad \hat{M}_t = \hat{m}_t = 1. \]

- Stochastic processes
  \[ \ln \lambda_{t+1} = \gamma \ln \lambda_t + \varepsilon^\lambda_t, \]
  and
  \[ \ln g_{t+1} = (1 - \pi) \ln \varphi + \pi \ln g_t + \varepsilon^\varphi_{t+1}. \]

Log-linear version of model

- The log-linear version of the first order conditions are
  \[ -\tilde{w}_t = \beta E_t \left[ \varphi (\tilde{r}_{t+1} - \tilde{w}_{t+1}) - (1 - \delta) \tilde{w}_{t+1} \right] \]
  and
  \[ -\frac{B}{mw} [\tilde{p}_t + \tilde{w}_t] = \beta E_t \left[ \frac{1}{\varphi} \tilde{g}_{t+1} \right] \]
  having used the cash in advance constraint in the form,
  \[ g_t \tilde{p}_t c^i_t = \hat{m}_{t-1} + g_t - 1. \]

- The flow budget constraint is
  \[ \kappa \tilde{k}_{t+1} + \frac{m}{\bar{p}} [\bar{m}_t - \bar{p}_t] = \bar{w} \bar{h} [\tilde{w}_t + \tilde{h}_t] + \tau \kappa [\tilde{r}_t + \tilde{k}_t] + (1 - \delta) \kappa \tilde{k}_t \]

Log-linear version of model

- Factor market conditions are
  \[ \tilde{r}_t = \tilde{K}^{\theta - 1} \tilde{H}^{1-\theta} \left[ \tilde{\lambda}_t + (\theta - 1) \left[ \tilde{K}_t - \tilde{H}_t \right] \right] \]
  and
  \[ \tilde{w}_t = \tilde{K}^\theta \tilde{H}^{-\theta} \left[ \tilde{\lambda}_t + \theta \left[ \tilde{K}_t - \tilde{H}_t \right] \right] \]
• The stochastic processes are

\[ \tilde{\lambda}_{t+1} = \gamma \tilde{\lambda}_t + \tilde{\varepsilon}_{\lambda t+1} \]

and

\[ \tilde{g}_{t+1} = \pi \tilde{g}_t + \tilde{\varepsilon}_{g t+1} \]

Getting rid of an annoying expectations

• One can remove the expectations from

\[ - \frac{B}{p^w} [\tilde{p}_t + \tilde{w}_t] = \beta E_t \left[ \frac{1}{y} \tilde{g}_{t+1} \right] \]

by using the process for money growth,

\[ \tilde{g}_{t+1} = \pi \tilde{g}_t + \tilde{\varepsilon}_{g t+1} \]

• Since the expectation of the error is zero, one can eliminate the expectations operator, and get

\[ - \frac{B}{p^w} [\tilde{p}_t + \tilde{w}_t] = \frac{\beta \pi \tilde{g}_t}{y} \]

The full model

• The equations without expectations are

\[ 0 = \bar{K} \bar{K}_{t+1} - \frac{1}{\beta} \tilde{p}_t - \bar{w} \bar{w}_t - \bar{w} \bar{H}_t - \bar{r} \bar{K} \bar{r}_t - \bar{r} \bar{K} \bar{H}_t - (1 - \delta) \bar{K} \bar{K}_t, \]

\[ 0 = \bar{r}_t - \bar{\lambda}_t - (\theta - 1) \bar{K}_t + (\theta - 1) \bar{H}_t, \]

\[ 0 = \bar{w}_t - \bar{\lambda}_t - \theta \bar{K}_t + \theta \bar{H}_t, \]

\[ 0 = \tilde{p}_t + \tilde{w}_t - \pi \tilde{g}_t \]

• one equation in expectations

\[ 0 = \tilde{w}_t + \beta \pi E_t \bar{r}_{t+1} - E_t \tilde{w}_{t+1}, \]

• two stochastic processes for the shocks to technology and money growth,

\[ \tilde{\lambda}_{t+1} = \gamma \tilde{\lambda}_t + \tilde{\varepsilon}_{\lambda t+1}, \]

\[ \tilde{g}_{t+1} = \pi \tilde{g}_t + \tilde{\varepsilon}_{g t+1}. \]

Solving the model

• The model can be written as
\[
\begin{align*}
0 & = Ax_t + Bx_{t-1} + Cy_t + Dz_t, \\
0 & = E_t [Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t], \\
    & \quad z_{t+1} = Nz_t + \varepsilon_{t+1},
\end{align*}
\]

where \( x_t = [K_{t+1}] \), \( y_t = \begin{bmatrix} \bar{r}_t \\ \bar{w}_t \\ \bar{H}_t \\ \bar{p}_t \end{bmatrix} \), and \( z_t = \begin{bmatrix} \bar{\lambda}_t \\ \bar{y}_t \end{bmatrix} \).

The matrices A to N are

\[
A = \begin{bmatrix} K \\ 0 \\ 0 \\ 0 \end{bmatrix},
\]

\[
B = \begin{bmatrix} -(\tau + 1 - \delta)K \\ (1 - \theta) \end{bmatrix},
\]

\[
C = \begin{bmatrix} -\tau K & -\bar{w}H & -\bar{w}H & -1 \\ 1 & 0 & (\theta - 1) & 0 \\ 0 & 1 & \theta & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix},
\]

\[
D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & \pi \end{bmatrix},
\]

\[
F = [0] \quad G = [0] \quad H = [0] \\
J = [\begin{bmatrix} \beta \tau & -1 & 0 & 0 \end{bmatrix} ] \\
K = [\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} ] \\
L = [\begin{bmatrix} 0 & 0 \end{bmatrix} ] \quad M = [\begin{bmatrix} 0 & 0 \end{bmatrix} ] \\
N = [\begin{bmatrix} \gamma & 0 \\ 0 & \pi \end{bmatrix} ]
\]

Solution of model

- We look for a solution of the form

\[
x_{t+1} = Px_t + Qz_t
\]

and

\[
y_t = Rx_t + Sz_t
\]
• Where

\[(F - JC^{-1}A)P^2 - (JC^{-1}B - G + KC^{-1}A)P - KC^{-1}B + H = 0,\]

and that

\[R = -C^{-1}(AP + B),\]

\[vec(Q) = (N' \otimes (F - JC^{-1}A) + I_k \otimes (FP + G + JR - KC^{-1}A))^{-1} \times vec((JC^{-1}D - L)N + KC^{-1}D - M),\]

and

\[S = -C^{-1}(AQ + D).\]

The solution matrices are

- \[P = \begin{bmatrix} 0.9418 \end{bmatrix}\]
- \[Q = \begin{bmatrix} 0.1552 & 0.0271 \end{bmatrix}\]
- \[R = \begin{bmatrix} -0.9450 & 0.5316 \\ -0.4766 & -0.5316 \end{bmatrix}\]
- \[S = \begin{bmatrix} 1.9418 & -0.0555 \\ 0.4703 & 0.0312 \\ 1.4715 & -0.0867 \\ -0.4703 & 0.4488 \end{bmatrix}\]

Variances

• How adding money shocks affect variances

<table>
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<th>(\sigma_\lambda = 0.0036)</th>
<th>(\sigma_\lambda = 0.0036)</th>
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<tr>
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Correlations with output
More money shocks reduce correlations with output

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Impulse response to technology shock
Response of Hansen model (no money) to tech shock
Response of Cooley-Hansen to money growth shock
Comments

- Note that variances and impulse response functions do not depend on the level of stationary state inflation
- Look at the first row of the A, B, C, D matrices
- All elements are divided by $\bar{g}$
- The relative values of this equation do not change with $\bar{g}$
- So the dynamic model does not change with $\bar{g}$
Figure 2: Response of Hansen’s model to technology shock

Figure 3: Response of Cooley-Hansen model to money growth shock
Seigniorage

- Alternative method of adding money to the economy
- Government consumes some goods
- Pays for these goods by issuing new money
- Budget constraint of the government is
  \[ g_t = \hat{g}_t \frac{M_t - M_{t-1}}{p_t} \]

  with the stochastic process
  \[ \ln \hat{g}_t = \pi \ln \hat{g}_{t-1} + \varepsilon_t^g \]

- Money issued depends on the real purchases of the government

Seigniorage

- Normalize by money stock at date \( t \)
- Government budget constraint becomes
  \[ g_t = \hat{g}_t \frac{M_t - M_{t-1}}{\frac{p_t}{\bar{p}_t}} = 1 - \frac{1}{\varphi_t} \]

  - Notation: Now \( \varphi_t = M_t/M_{t-1} \)
    - It is the gross growth rate of money (NOT \( g_t \))

Seigniorage

- Rest of model: household optimization problem
  \[ E_0 \sum_{t=0}^{\infty} \beta^t u(c_t^i, 1 - h_t^i), \]

- subject to the sequence of cash in advance constraints,
  \[ \hat{p}_t c_t^i \leq \frac{\hat{m}_{t-1}^i}{\varphi_t}, \]

- the sequence of family real budget constraints,
  \[ k_{t+1}^i + \frac{\hat{m}_t^i}{\bar{p}_t} = w_t h_t^i + r_t k_t^i + (1 - \delta)k_t^i. \]
Seigniorage
• the economy wide cash in advance constraints (at equality) are
\[ p_t C_t + p_t \bar{g}_t = p_t C_t + p_t \bar{g}_t \bar{g} = M_t, \]
or
\[ \hat{p}_t C_t + \hat{p}_t \bar{g}_t \bar{g} = 1, \]
• The cash in advance for the households is
\[ p_t C_t = M_{t-1}, \]
• dividing both sides of this equation by \( M_t \),
\[ \hat{p}_t C_t = \frac{1}{\bar{\varphi}_t}, \]
• The real budget constraint for the economy is
\[ C_t + K_{t+1} + \bar{g}_t \bar{g} = w_t H_t + r_t K_t + (1 - \delta) K_t, \]
Seigniorage
• Competitive factor markets imply that
\[ r_t = \theta \lambda_t K_t^{\theta-1} H_t^{1-\theta}, \]
and
\[ w_t = (1 - \theta) \lambda_t K_t^{\theta} H_t^{-\theta}, \]
Seigniorage
• First order conditions are
\[ \frac{1}{w_t} = \beta E_t \left[ \frac{r_{t+1} + 1 - \delta}{w_{t+1}} \right] \]
• and
\[ -\frac{B}{\hat{p}_t w_t} = \frac{\beta}{\bar{m}_t}. \]
Seigniorage: Stationary state
• From FOCs
\[ \frac{1}{\bar{\beta}} = \pi + (1 - \delta) \]
and
\[ \frac{\beta \bar{\pi}}{B} = \frac{\hat{\bar{m}}}{\bar{\bar{p}}}. \]
• From factor market
\[
\bar{w} = (1 - \theta) \left[ \frac{\theta}{\frac{1}{\beta} - (1 - \delta)} \right]^{\frac{\theta}{\varphi}}
\]

• From government budget constraint
\[
\gamma = 1 - \frac{1}{\varphi}
\]

• Some algebra gives
\[
\varphi = \frac{\beta \bar{w}}{\beta \gamma + \beta \bar{w}}
\]

Seigniorage

• Bailey curve (example economy)

Seigniorage: log-linear version

• Model is
\[
\begin{align*}
0 & = \bar{w}_t + \tau \beta E_t \bar{r}_{t+1} - E_t \bar{w}_{t+1}, \\
0 & = -\bar{w}_t + \bar{p}_t, \\
0 & = \gamma \bar{p} [\bar{p}_t + \bar{g}_t] - \frac{1}{\varphi} \bar{\varphi}_t, \\
0 & = K_{t+1} - \frac{1}{\rho} \bar{p}_t - \pi \pi [\bar{w}_t + \bar{H}_t] - \tau \pi K [\bar{r}_t + K_t] - (1 - \delta) K K_t, \\
0 & = \bar{r}_t - \bar{\lambda}_t - (\theta - 1) K_t - (1 - \theta) \bar{H}_t, \\
0 & = \bar{w}_t - \bar{\lambda}_t - \theta K_t + \theta \bar{H}_t.
\end{align*}
\]
• Plus the stochastic processes for technology and government expenditures

Seigniorage: log-linear version

• Define variables as
  \[ x_t = \begin{bmatrix} \tilde{K}_{t+1} \\ \bar{w}_t \\ \bar{p}_t \\ \bar{\sigma}_t \\ H_t \end{bmatrix}, \quad y_t = \begin{bmatrix} \tilde{g}_t \\ \hat{\lambda}_t \end{bmatrix}, \quad z_t = \begin{bmatrix} \tilde{y}_t \end{bmatrix}, \]

• write the model as we did earlier

\[
0 = Ax_t + Bx_{t-1} + Cy_t + Dz_t, \\
0 = E_t [Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t], \\
z_{t+1} = \frac{\bar{\sigma}_t}{Nz_t + \varepsilon_{t+1}}.
\]

• Solve for
  \[ x_{t+1} = Px_t + Qz_t \]
  and
  \[ y_t = Rx_t + S z_t. \]

Seigniorage: results

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