CHAPTER 33

VALUING BONDS

The value of a bond is the present value of the expected cash flows on the bond, discounted at an interest rate that is appropriate to the riskiness of that bond. Since the cash flows on a straight bond are fixed at issue, the value of a bond is inversely related to the interest rate that investors demand for that bond. The interest rate charged on a bond is determined by both the general level of interest rates, which applies to all bonds and financial investments, and the default premium specific to the entity issuing the bond. This chapter examines the determinants of both the general level of interest rates and the magnitude of the default premia on specific bonds. The general level of interest rates incorporates expected inflation and a measure of real return and reflects the term structure, with bonds of different maturities carrying different interest rates. The default premia varies across time, depending in large part on the health of the economy and investors' risk preferences.

Bonds often have special features embedded in them that have to be factored into the value. Some of these features are options - to convert into stock (convertible bonds), to call the bond back if interest rates go down (callable bonds) and to put the bond back to the issuer at a fixed price under specific circumstances (putable bonds). Other bond characteristics, such as interest rate caps and floors, have option features. Some of these options reside with the issuer of the bond, some with the buyer of the bond, but they all have to be priced. Option pricing models can be used to value these special features and price complex fixed income securities. Some special features in bonds such as sinking funds, subordination of further debt and the type of collateral may affect the prices of bonds, as well.

Bond Prices and Interest Rates

The value of a straight bond is determined by the level of and changes in interest rates. As interest rates rise, the price of a bond will decrease and vice versa. This inverse relationship between bond prices and interest rates arises directly from the present value relationship that governs bond prices.

a. The Present Value Relationship

The value of a bond, like all financial investments, is derived from the present value of the expected cash flows on that bond, discounted at an interest rate that reflects the default risk associated with the cash flows. There are two features that set bonds apart from equity investments. First, the cash flows on a bond, i.e., the coupon payments and the face value of the bond, are usually set at issue and do not change during the life of the bond. Even when they do change, as in floating rate bonds, the changes are generally linked to changes in interest rates. Second, bonds usually have fixed lifetimes, unlike stocks, since most bonds[1] specify a maturity date. As a consequence, the present value of a 'straight bond' with fixed coupons and specified maturity is determined entirely by changes in the discount rate, which incorporates both the general level of interest rates and the specific default risk of the bond being valued.

The present value of a bond, expected to mature in N time periods, with coupons every period can be calculated.

$$\text{PV of Bond} = \sum_{t=1}^{t=N} \frac{\text{Coupon}_t}{(1+r)^t} + \frac{\text{Face Value}}{(1+r)^N}$$

where,

Coupon_t = Coupon expected in period t

Face Value = Face value of the bond

r = Discount rate for the cash flows

The discount rate used to calculate the present value of the bond will vary from bond to bond depending upon default risk, with higher rates used for riskier bonds and lower rates for safer ones.

If the bond is traded, and a market price is therefore available for it, the internal rate of return can be computed for the bond, i.e., the discount rate at which the present value of the coupons and the face value is equal to the market price. This internal rate of return is called the yield to maturity on the bond.

[1] Console bonds are the exception to this rule, since they are perpetuities.

There are several details, relating to both the magnitude and timing of cash flows, that can affect the value of a bond and its yield to maturity. First, the coupon payment on a bond may be semi-annual, in which case the discounting has to allow for the semi-annual cash flows. (The first coupon will be discounted back half a year, the second one year, the third a year and a half and so on.) Second, once a bond has been issued, it accrues coupon interest between coupon payments and this accrued interest has to be added on to the price of the bond, when valuing the bond.

Illustration 33.1: Valuing a straight bond at issue

The following is a valuation of a thirty-year U.S. Government Bond at the time of issue. The coupon rate on the bond is 7.50%, and the market interest rate is 7.75%. The price of the bond can be calculated.

$$\text{PV of Bond} = \sum_{t=1}^{t=30} \frac{75.00}{(1.0775)^t} + \frac{1,000}{(1.0775)^{30}} = \$971.18$$

This is based upon annual coupons. If the calculation is based upon semi-annual coupons, the value of the bond is:

$$\text{PV of Bond} = \sum_{t=0.5}^{t=30} \frac{37.50}{(1.0775)^t} + \frac{1,000}{(1.0775)^{30}} = \$987.62$$

Illustration 33.2: Valuing a seasoned straight bond

The following is a valuation of a seasoned Government bond, with twenty years left to expiration and a coupon rate of 11.75%. The next coupon is due in two months. The current twenty-year bond rate is 7.5%. The value of the bond can be calculated.

$$\text{PV of Bond} = \sum_{t=0.5}^{t=19.5} \frac{58.75}{(1.075)^t} + \frac{58.75}{(1.075)^{2/12}} + \frac{1,000}{(1.075)^{19.67}} = \$1505.31$$

This bond trades at well above face value, because of its high coupon rate. Note that the second term of the equation is the present value of the next coupon.

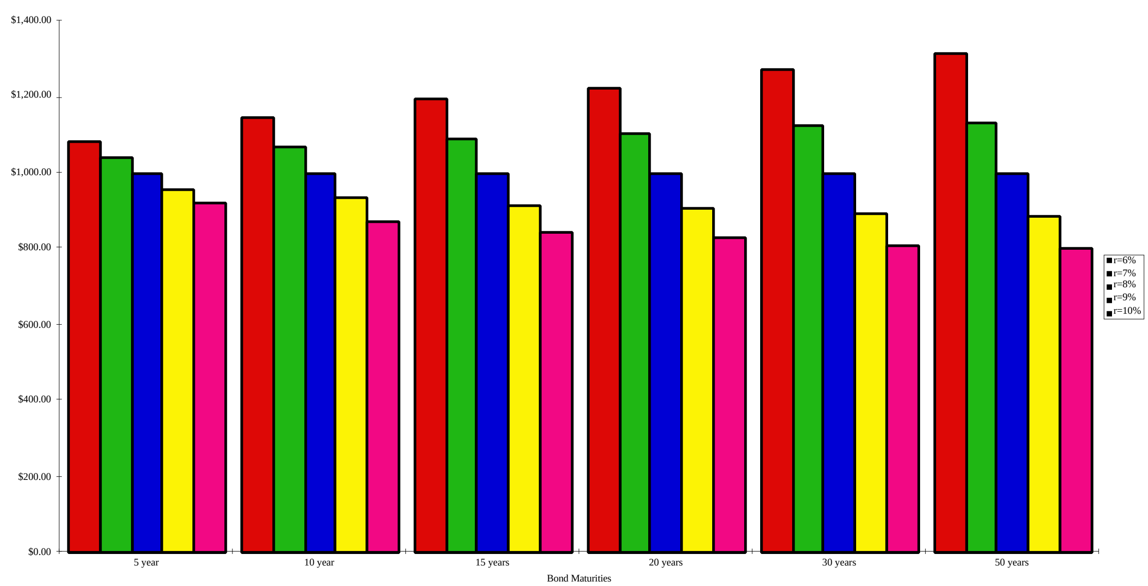
b. A Measure of Interest Rate Risk in Bonds

When the fact that the cash flows on a bond are fixed at issue is combined with the present value relationship governing bond prices, there is a clear rationale for why interest changes affect bond prices so directly. Any increase in interest rates, either at the economy wide level or because of an increase in the default risk of the company issuing the bond, will lower the present value of the stream of expected cash flows and hence the value of the bond. Any decrease in interest rates will have the opposite impact.

The effect of interest rate changes on bond prices will vary from bond to bond and will depend upon a number of characteristics of the bond.

(a) the maturity of the bond - Holding coupon rates and default risk constant, increasing the maturity of a straight bond will increase its sensitivity to interest rate changes. The present value of cash flows changes much more for cash flows further in the future, as interest rates change, than for cash flows which are nearer in time. Figure 33.1 illustrates the present values of six bonds - a 5-year, a 10-year, a 15-year, a 20-year, a 30-year and a 50-year bonds, all with 8% coupons for a range of interest rates.

Figure 33.1: Bond Values and Interest Rates

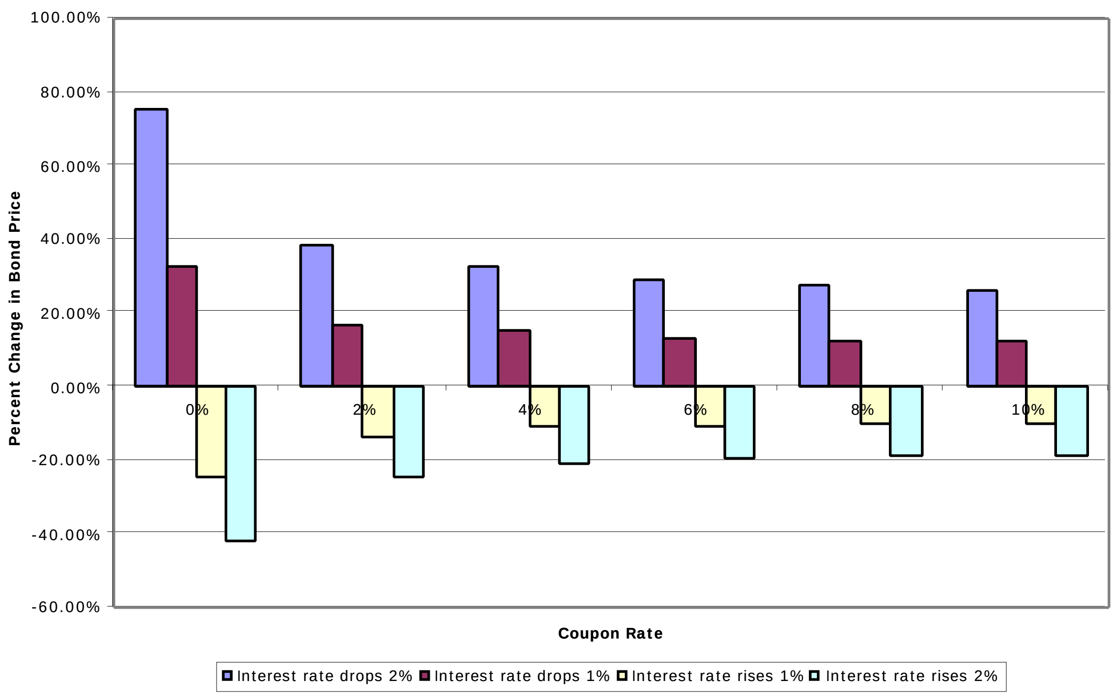


The longer-term bonds are much more sensitive to interest rate changes than the shorter term bonds. For instance, an increase in interest rates from 8% to 10% results in a decline in value of 7.61% for the five-year bond and of 19.83% for the fifty-year bonds.

(b) the coupon rate of the bond - Holding maturity and default risk constant, increasing the coupon rate of a straight bond will decrease its sensitivity to interest rate changes. Since higher coupons result in more cash flows earlier in the bond's life, the present value will

change less as interest rates change. At the extreme, if the bond is a 'zero-coupon' bond, the only cash flow is the face value at maturity, and the present value is likely to vary much more as a function of interest rates. Figure 33.2 illustrates the percentage changes in bond prices for six thirty-year bonds with coupon rates ranging from 0% to 10% for a range of interest rates.

Figure 33.2: Percent Change in Bond Price - Interest rate changes from 8%



The bonds with the lower coupons are much more sensitive, in percentage terms, to interest rate changes than those with higher coupons.

While the maturity and the coupon rate are the key determinants of how sensitive the price of a bond is to interest rate changes, a number of other factors impinge on this sensitivity. Any special features that the bond has, including convertibility and callability, make the maturity of the bond less definite and can therefore affect the bond price's sensitivity to interest rate changes. If there is any relationship between the level of interest rates and the default premia on bonds, the default risk of a bond can affect its price sensitivity.

c. A More Formal Measure of Interest Rate Risk - Duration

Since the interest rate risk of a bond is a significant component of its total risk, a more formal measure of interest risk is needed, which consolidates the effects of maturity, coupon rates and the bond's special features. To arrive at this measure, consider the present value relationship developed earlier in this chapter.–

$$\text{PV of Bond} = \sum_{t=1}^{t=N} \frac{\text{Coupon}_t}{(1+r)^t} + \frac{\text{Face Value}}{(1+r)^N}$$

Differentiating the bond price with respect to interest rate should provide a formal measure of bond price sensitivity to interest rate changes.

$$\text{Duration of Bond} = \frac{\text{dP/P}}{\text{dr/r}} = \frac{\left[\sum_{t=1}^{t=N} \frac{t * \text{Coupon}_t}{(1+r)^t} + \frac{N * \text{Face Value}}{(1+r)^N}\right]}{\left[\sum_{t=1}^{t=N} \frac{\text{Coupon}_t}{(1+r)^t} + \frac{\text{Face Value}}{(1+r)^N}\right]}$$

The bond price differential, $\frac{dP/P}{dr/r}$, is called the duration of the bond and measures the interest rate sensitivity of the bond.

The duration of a bond is a weighted maturity of all the cash flows on the bond including the coupons, where the weights are based upon both the timing and the magnitude of the cash flows. Larger and earlier cash flows are weighted more than smaller and later cash flows. By incorporating the magnitude and timing of all the cash flows on the bond, duration encompassed all the variables that affect bond price sensitivity in one measure. The higher the duration of a bond, the more sensitive it is to changes in interest rates.

The duration of a bond will always be less than the maturity for a coupon bond and equal to the maturity for a zero-coupon bond, with no special features. In general, the duration of a bond will decrease as the coupon rate on the bond increases.

The measure of duration described here is called 'Macaulay duration' and it is the simplest version, based upon yields to maturity. It is based upon the assumption of a flat term structure and modified versions of duration, which are more flexible in their assumptions about the term structure and its shifts over time

Illustration 33.3: Estimating durations for coupon bonds

In this example, we estimate the duration of a seasoned Government bond, with twenty years left to expiration and a coupon rate of 11.75%. The interest rate is 7.5%. The duration of the bond, assuming annual coupon payments, can be calculated as follows.

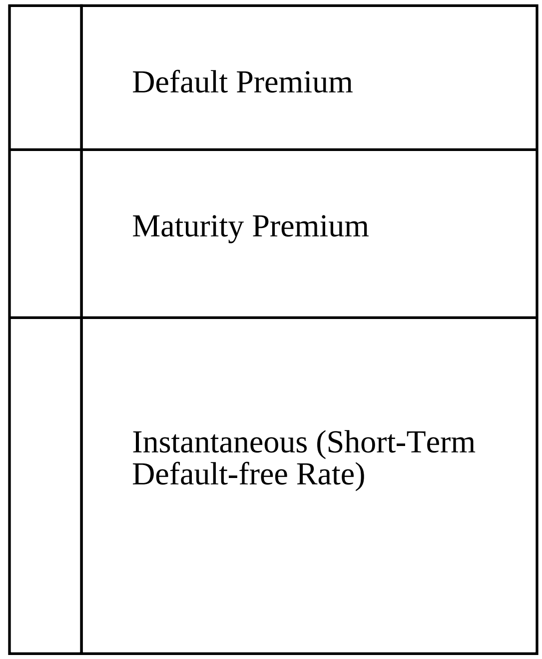
t	*Cashflow*	*PV of Cashflow*	*t * PV of Cashflow*
1	$117.50	$109.30	$109.30
2	$117.50	$101.68	$203.35
3	$117.50	$94.58	$283.75
4	$117.50	$87.98	$351.94
5	$117.50	$81.85	$409.23
6	$117.50	$76.14	$456.81
7	$117.50	$70.82	$495.77
8	$117.50	$65.88	$527.06
9	$117.50	$61.29	$551.57
10	$117.50	$57.01	$570.10
11	$117.50	$53.03	$583.36
12	$117.50	$49.33	$591.99
13	$117.50	$45.89	$596.58
14	$117.50	$42.69	$597.65
15	$117.50	$39.71	$595.67
16	$117.50	$36.94	$591.05
17	$117.50	$34.36	$584.17
18	$117.50	$31.97	$575.38
19	$117.50	$29.74	$564.98
20	$1,117.50	$263.07	$5,261.48
		$1,433.27	$14,501.21

$$\text{Duration of the Bond} = \frac{\$14{,}501}{\$1{,}433} = 10.12$$

Determinants of Interest Rates

The discount rate used to discount cash flows on a bond is determined by a number of variables - the general level of interest rates in the economy, the term structure of interest rates and the default risk of the bond. Figure 33.3 provides the building blocks for arriving at the interest rate on a straight corporate bond.

Figure 33.3: Building Blocks for Interest Rates



The first block is the level of short-term default free interest rates and it captures the overall level of rates in the economy. The second block is a maturity premium, which reflects the difference between longer-term default free rates and short-term default free rates, and is generally positive. The third block is a default premium, which is related to the default risk of the bond is question. This section takes a closer look at these blocks.

a. Level of Interest Rates

The short-term default free rate can be decomposed into two components - an expected inflation rate during the period and an expected real rate of return.

Short-term default free rate = Expected Inflation + Expected Real Rate of Return

This identity is known as the Fisher equation and essentially implies that changes in short-term rates can be traced to changes in either expected inflation or the expected real rate of return. The more precise version of the Fisher equation allows for the compounding effect.

$$(1+r) = (1+I)\,(1+R)$$

where,

r = Nominal interest rate

I = Expected Inflation

R = Expected real rate of return

It should be emphasized that the Fisher equation is an identity and there is no question of it being proved or disproved. The real questions that arise from the equation are the specific assumptions about the real rate and expected inflation.

I. Expected Inflation

Expected inflation is clearly the dominant variable determining interest rates. Generally speaking, a forecaster who can predict changes in inflation well should also post a good track record in predicting interest rate changes. The first step in forecasting inflation is the understanding of its determinants.

The Determinants of Inflation

There is consensus on the determinants of inflation, though there is little agreement about the consequences of specific actions on inflation. To understand both the determinants of inflation and the sources of disagreement between the different schools of thought on inflation, consider another identity.

$$P = \frac{MV}{Y}$$

Where

P = Price level

M = Money supply in the economy

V = Velocity of money circulation in the economy

Y = Real Output in the economy

The velocity of money measures how often the currency, used to define the money supply 'M', circulates in the economy and how much is created in terms of transactions for every unit of currency created. Thus, if a $1 in additional currency created $3 in transactions, the velocity of money is 3. While the money supply used in the equation can be defined in a number of different ways, ranging from just currency to broader aggregates, the velocity has to be defined consistently.

This identity can be stated in terms of changes as follows -

$$dP = \frac{dM\,dV}{dY}$$

The left hand side of this identity is the inflation rate and the right hand side provides the three determinants of the inflation rate.

(a) the change in the money supply (dM): If the money supply increases, with no concurrent change in real output and money velocity, the inflation rate will increase. This is the basis for the argument by many monetarists, who believe that there is no linkage between real output and money supply and that money velocity is stable over long periods, that loose monetary policy (increasing money supply) is the reason for high inflation. While some monetarists will concede that monetary policy can have short term effects on real output, most argue that it cannot impact real output in the long term. They also argue that while money velocity may change over time, that these changes occur over the very long term and are unlikely to have a major impact on inflation.

(b) the change in money velocity (dV): If the money velocity increases, with no concurrent change in money supply and real output, the inflation rate will increase. Economists have long debated why money velocity changes over time. One determinant is technology, since changes in the way people save (from checking accounts to money market accounts) and in the way they spend (from cash transactions to credit card transactions) affect the money velocity. Another is the faith the public has in the currency. In hyper-inflationary environments, individuals are much less willing to hold currency (because it depreciates in value so quickly) and therefore attempt to convert the currency into real goods. This unwillingness to hold currency translates into higher money velocity. Thus, if the central bank is viewed as having eased the reins on money supply, there is often a concurrent increase in money velocity, leading to a surge in inflation.

(c) the change in real output: If the real output increases, with no concurrent change in money supply and money velocity, the inflation rate will decrease. This is often the basis of the argument used by Keynesians for easing monetary policy during economic downturns. Increasing the money supply, they argue, results in a concomitant increase in real output, since there is excess capacity, and the effects on inflation are therefore muted or non-existent.

Measuring Inflation

A true measure of inflation would consider changes in the prices of all goods and services used in an economy, weighted by their usage values. The reported measures of inflation, either at the consumer or the producer level, attempt to do so, but often lag changes in true inflation because of a number of reasons. The first is that not all goods and services are traded in a market place, where prices are easily available and goods are fairly standardized. Thus, it is easy to gauge the inflation in medical prescription prices, but much more difficult to gauge the inflation in the prices of medical services. The second is that all inflation indices are based upon samplings of prices of goods, rather than the universe of all goods traded. Even if the sample is not biased, there is the possibility of sampling error that enters into the numbers. The third is the issue of weighting on the basis of usage value. Due to practical considerations of time and resources, the weights are not adjusted every time the inflation index is computed to allow for changes in usage. Instead index weights are adjusted infrequently, leading to biased in the measured inflation. Thus, the inflation indices which kept the usage of gasoline by households constant in the late seventies while oil prices were climbing (and people were cutting back on the use of gasoline) tended to overstate the inflation rate during that period. The final consideration is about the level at which inflation is to be measured, since counting goods at every level of the process (from commodity to manufactured good to retailed good) would result in double or even triple counting the same good. Different inflation indices examine inflation at different stages in the process and can lead to different conclusions about whether inflation is increasing, decreasing or staying unchanged.

Forecasting Inflation

Since changes in inflation signal changes in interest rates, economists and analysts have expended considerable time and resources forecasting inflation, with mixed results. The forecasting models used range from the naive to the sophisticated and are based upon everything from gut feeling to elaborate mathematics. The output from these models can be contrasted with predictions based purely upon past inflation - either the inflation in the last time period or time-series models that examine trends and shifts in past inflation - and the results for the most part are mixed. Elaborate forecasting models do no better than time series models in the short term, but may better capture changes in inflation in the long term because they consider information beyond what's available in past inflation rates.

The introduction of inflation-adjusted treasury bonds a few years ago has provided an interesting alternative for those who would rather rely on markets for their inflation estimates than economists. In particular, if we view the market interest rate on an inflation indexed treasury bond as a riskless real rate and the market interest rate on a nominal treasury bond of equal maturity as a nominal rate, the expected inflation rate can be estimated as follows:

$$\text{Expected Inflation Rate} = \frac{(1+\text{Nominal Rate})}{(1+\text{ Real Rate})} - 1 = \text{Expected Inflation Rate}$$

For instance, if the nominal rate is 5.1% and the real rate is 2.7%, you can estimate the expected inflation rate as follows:

Expected Inflation Rate = 1.051/1.027 = .0233 or 2.33%

Testing the Fisher Equation

As mentioned earlier, the Fisher equation is an identity that cannot be proved or disproved. There have, however, been numerous attempts to impose additional constraints on the model, to test the usefulness of the model in explaining changes in interest rates over time. These studies go back to Fisher's own work on interest rates and inflation, where he found that the correlation between the rate of inflation and the commercial paper rate was low in both his sample periods – 1890 to 1914 and 1915 to 1927. The correlation between inflation and the commercial paper rate did not improve as various leads and lags on inflation were tried.

Fama (1976) made the assumption that real rates do not change much over time and that changes in interest rates should therefore almost entirely be caused by changes in inflation. He tested this proposition by regressing interest rates against expected inflation.

$I_t = a + b\, R_t$

where,

R_t = Nominal interest rate during period t

I_t = Expected inflation during period t

He argued that if his initial assumption about constant real rates was true, this regression would yield the following:

(a) The intercept would be equal to the constant real rate over the period.

(b) The slope of the regression would be one, since all changes in interest rates would be a consequence of changes in inflation.

Lacking an adequate measure of expected inflation, he used the one-month treasury bill rate at the start of each month as a measure of expected inflation during the month and the one and three-month treasury bill rates as measures of nominal rates. His results, for the period 1953 to 1971, were as follows -

CPI regressed against one-month T.Bills

$I_t =$	0.0007 -	0.98 R_t	$R^2 = 0.29$
	(0.0003)	(0.10)	

CPI regressed against three-month T.Bills

$I_t =$	0.0023 -	0.92 R_t	$R^2 = 0.48$
	(0.0011)	(0.11)	

Based upon this regression, he concluded that the hypothesis of constant real rates was supported and that the slope was statistically indistinguishable from one, suggesting that there was a one-to-one relationship between changes in interest rates and expected inflation.

The studies that followed up have generally not been as encouraging. Wood, for instance, updates Fama's regression, after adding a lagged measure of inflation and contrasts the results for two periods – 1953 to 1971 and 1974 to 1981.

$$I_t = a + b\, R_t + c\, I_{t-1}$$

Period	*Regression*			*R^2*
1953-71	I_t = 0.0006	- 0.84 R_t	+ 0.09 I_{t-1}	0.309
	(0.0003)	(0.111)	(0.064)	
1974-81	I_t = - 0.0023	- 0.25 R_t	+ 0.47 I_{t-1}	0.371
	(0.0008)	(0.12)	(0.11)	

The coefficient on nominal interest rates (R_t) which was close to one for the 1953-71 time period, used by Fama in his study, drops to 0.25 for the 1974-81 time period.

The reason for the surprisingly good results from 1953 to 1971 may be traceable to the fact that inflation was very stable during this period and that changes in inflation tended to be small. Thus, it seems likely that the hypothesis of stable real rates and a one-to-one relationship between interest rates and inflation will be rejected in any period or any economy where there is volatility in interest rates and inflation. Since the importance of forecasting increases with the volatility of interest rates and inflation, the cautionary notes on forecasting short-term interest rates based only upon expected inflation should be taken to heart.

II. Expected Real Rate of Return

The other component of the Fisher equation is the expected real rate of return. On an intuitive level, the expected real rate of return is the rate at which individuals are willing to trade off current consumption for future consumption. Given the human preference for present consumption, the expected real rate of return should be positive, but can vary widely across time and across economies. If individuals in a society have a strong desire for current consumption, the expected real rate of return will have to be high to induce them to defer consumption.

a. Realized Real Rates of Return

Since the expected real rate of return is based upon the preference functions of individuals, which are difficult to observe, we are reduced to observing realized real rates of return, which can be defined to be –

$$\text{Realized Real Rate of Return} = \text{Nominal Interest Rate}_t - \text{Actual Inflation}_t$$

where,

$\text{Nominal Interest Rate}_t$ = Nominal interest rate at the beginning of period t

$\text{Actual Inflation}_t$ = Actual Inflation during period t

While the expected real rate of return should be positive, the realized real rate of return can be positive or negative, depending upon the period under observation. During the 1970s, for instance, bond investors in the United States earned negative real rates of return as actual inflation outstripped expected inflation.

b. Expected Real Return and Expected Real Growth

Ultimately, real returns to investors in an economy comes from real growth in the economy. One way to approach the estimation of expected real return is to estimate the expected real growth rate in the economy. Thus, the expected real return in an economy growing in the long term at 2.5% a year, should be approximately 2.5%. If the expected real return increases above the long term growth rate in the economy, the imbalance will lead to a depletion of savings and a shortfall in investments. Alternatively, if the real return decreases below the long term growth rate, the imbalance will lead to an accumulation of savings and over-investment.

The Role of the Central Bank

Central banks do not set interest rates, but they certainly can influence them in two ways. On a short term basis, central banks can tighten or loosen its reins on the money supply and try to slow an overheated economy or regenerate a sluggish economy. In either case, though, we should not attribute more power to central banks than they actually have. The only interest rate that the Federal Reserve in the United States, for instance, directly controls is the Federal funds rate. By raising or lowering this rate, it can hope to affect other rates but the market does not always cooperate. It is generally true that market interest rates tend to move with the Federal funds rate, but there are two caveats. The first is that markets tend to lead the Federal reserve as bond investors build in expectations of changes in Fed policy. And the second is that the correlation tends to be strongest for short term rates (treasury bills and commercial paper) and weaker for long maturity bonds.

On a long-term basis, central banks can have a much bigger impact on interest rates through their conduct of monetary policy and the resolution that they show about fighting inflation. It is no coincidence that high inflation occurs most often when central banks are undisciplined when it comes to monetary policy and show no resolve when it comes to taking tough measures to fight inflation.

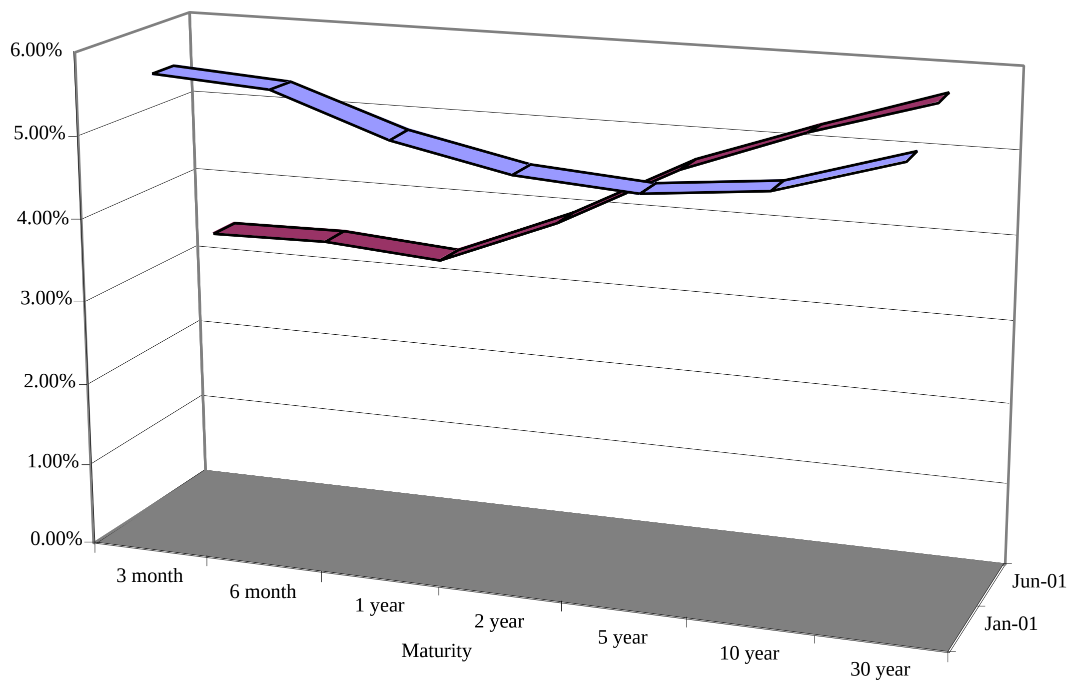
b. Maturity Premium

The maturity premium refers to the difference in interest rates between a short-term (or instantaneous) default-free bond and a longer-maturity default-free bond. In the following section, the maturity premium is clarified further and a number of different theories designed to explain the magnitude of the maturity premium are examined.

a. The Yield Curve

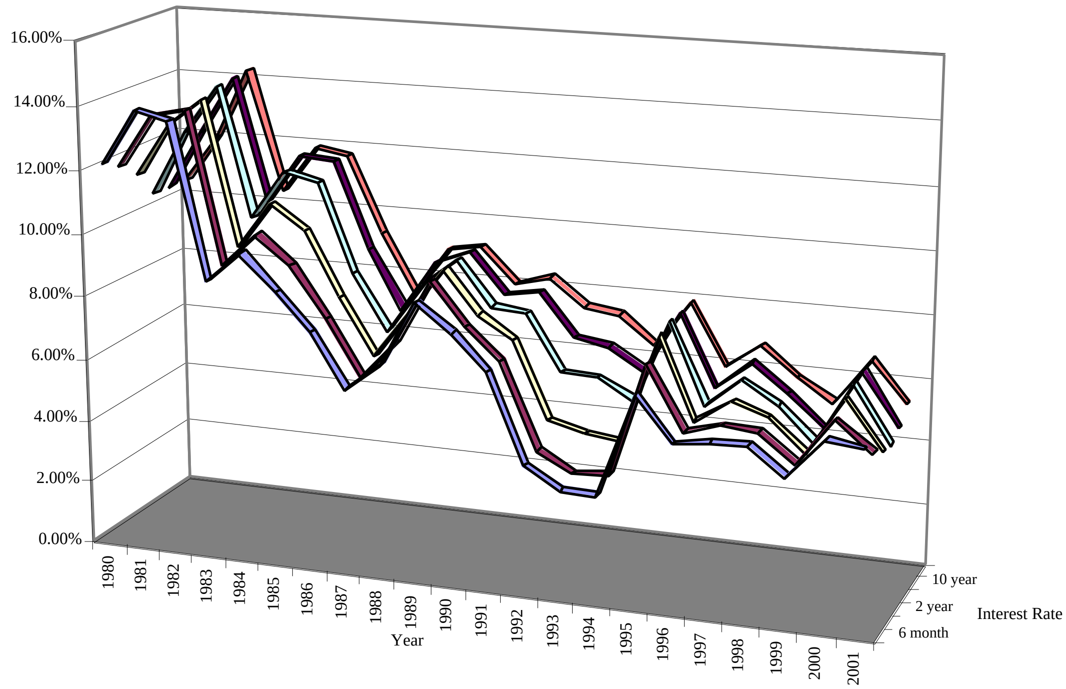
The relationship between maturity and interest rates is usually captured by a yield curve, which graphs yields on bonds against bond maturities. Figure 33.4 summarizes the treasury yield curve in January and June 2001.

Figure 33.4: Yield Curves - January 2001 and June 2001



In January 2001, the yield curve was slightly downward sloping. But by June 2001, the yield curve had reverted – short term rates dropped while long term rates increased slightly. While the yield curve has generally been upward sloping over much of this century, there have been periods where the yield curve has been downward sloping. Figure 33.5 shows the yield curves from 1980 to 2001.

Figure 33.5: Yield Curves : 1980-2001



In the early 1980s, short term rates were higher than long term rates for a period. Over the last two decades, rates have dropped at both ends of the spectrum.

While the yield curves are generally constructed using the yields to maturity of government bonds, the presence of coupons on these bonds affects the calculated yield to maturity. This limitation can be overcome in one of two ways. The first is to construct a yield curve using only zero coupon government coupon bonds of different maturity. The second is to extract spot interest rates from the yields to maturity of coupon bonds and to plot the spot rates against maturities. The following example illustrates the process of extracting spot rates.

Illustration 33.4: Yields to Maturity and Spot Rates

The following table provides prices and yields to maturity on one to five year bonds and extracts spot rates from the yields to maturity.

Maturity	Yields to Maturity	Spot Rate
1 year	4.00%	4.00%
2 year	4.25%	4.26%

3 year	4.40%	4.41%
4 year	4.50%	4.514%
5 year	4.58%	4.60%

The spot rate is estimated from the two year rate as follows –

Price of two year bond $= \dfrac{\text{Coupon}_1}{1+{}_0r_1} + \dfrac{\text{Face Value} + \text{Coupon}_2}{\left(1+{}_0r_2\right)^2}$

Assuming the bond is priced at par,

$$1000 = \frac{42.50}{1.04} + \frac{1042.50}{\left(1+{}_0r_2\right)^2}$$

Solve for ${}_0r_2$.

$${}_0r_2 = \left(\frac{1042.50}{1000 - \frac{42.50}{1.04}}\right)^{0.5} - 1 = 4.26\%$$

The other rates are extracted using a similar process,

$$1000 = \frac{44.00}{1.04} + \frac{44.00}{1.0426^2} + \frac{1044.00}{\left(1+{}_0r_3\right)^3}$$

$$1000 = \frac{45.00}{1.04} + \frac{45.00}{1.0426^2} + \frac{45.00}{1.0441^3} + \frac{1045.00}{\left(1+{}_0r_4\right)^4}$$

$$1000 = \frac{45.80}{1.04} + \frac{45.80}{1.0426^2} + \frac{45.80}{1.0441^3} + \frac{45.80}{1.0451^4} + \frac{1045.80}{\left(1+{}_0r_5\right)^5}$$

The difference between yields to maturity and spot rates increases as the bond maturity increases.

b. Spot and Forward Rates

The spot rate on a multi-period bond is an average rate that applies over the periods. The forward rate is a one-period rate for a future period and can be extracted from the spot rates. For instance, if ${}_0S_2$ is the two-period spot rate and ${}_0S_1$ is the one-period spot rate, the forward rate for the second period, ${}_1F_2$, can be obtained.

$${}_1F_2 = \frac{\left(1+{}_0S_2\right)^2}{1+{}_0S_1}$$

The forward rate for period three can be extracted using the spot rates for periods 2 and 3. In general, the forward rate for period n can be written as:

$$_{n-1}F_n = \frac{(1+{_0}S_n)^n}{(1+{_0}S_{n-1})^{n-1}}$$

If the yield curve for spot rates is upward sloping, the yield curve using forward rates will be even more so. Alternatively, if the spot rate yield curve is downward sloping, the forward rate yield curve will be even more so. The following illustration builds on the previous one and extracts forward rates from spot rates.

Illustration 33.5: Spot Rates and Forward Rates

The forward rates are extracted from the spot rates for one to five year bonds. This is illustrated in the following table.

	YTM	*Spot Rate*	*Forward Rate*
1	4.00%	4.00%	4.00%
2	4.25%	4.26%	4.52%
3	4.40%	4.41%	4.71%
4	4.50%	4.51%	4.81%
5	4.58%	4.60%	4.96%

Forward rate for year 2 $= \frac{1.0426^2}{1.04} - 1 = 4.52\%$

Forward rate for year 3 $= \frac{1.0441^3}{1.0426^2} - 1 = 4.71\%$

Forward rate for year 4 $= \frac{1.0451^4}{1.0441^3} - 1 = 4.81\%$

Forward rate for year 5 $= \frac{1.0460^5}{1.0451^4} - 1 = 4.96\%$

c. Determinants of the Maturity Premium

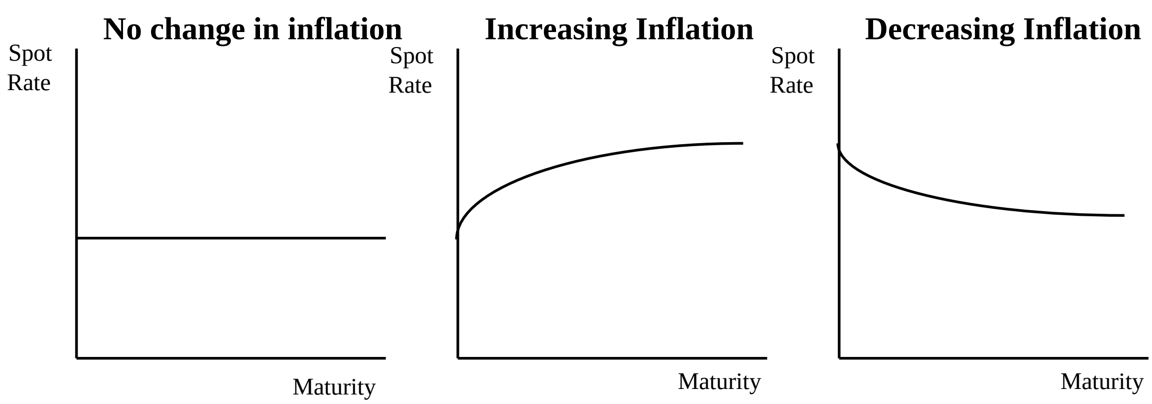
The magnitude of the maturity premium is determined by a number of factors including expectations about inflation, investor preferences for liquidity and demands from specific market segments. Each of these factors is examined in more detail in the following section.

1. Expected Inflation

Expectations about future inflation are a key determinant of longer term rates. In general, if inflation is expected to go up in future periods, longer term rates will be higher than shorter term rates. Alternatively, if inflation is expected to go down in future period, longer term rates will be lower than short term rates.

An extreme version of this story is the 'pure expectations hypothesis', where the term structure is driven entirely by the expectations on inflation. Under this hypothesis, the yield curve will be upward sloping if investors expect inflation to rise in future periods, flat if investors expect inflation to remain unchanged in future periods, and downward sloping if investors expect inflation to decline in future periods. This is illustrated in Figure 33.6.

Figure 33.6: Pure Expectations Hypothesis



The pure expectations hypothesis can also be stated in terms of forward rates and expected spot rates. If the hypothesis is correct, the forward rate for period n should be the best predictor of the expected spot rate in that period.

$$_{n-1}F_n = \text{Exp}(_{n-1}S_n)$$

where,

$_{n-1}F_n$ = Forward rate for period n

$\text{Exp}(_{n-1}S_n)$ = Expected one-period spot rate in period n

While the pure expectations hypothesis may be extreme in assuming that forward rates are determined entirely by expected spot rates, it does highlight the importance of expected inflation in determining the maturity premium.

2. Liquidity Preference

The liquidity preference theory is not an alternative to the expectations theory. It builds on expectations by taking into account uncertainty and risk aversion. In the form in which it was originally developed by Hicks (1946), the uncertainty was seen as accruing to the lender who concurrently charged a liquidity premium for lending for longer time periods. This uncertainty can also be stated in terms of bond prices, with long term bonds being viewed as more volatile than short term bonds, as interest rates change. Under this theory, holding expectations of inflation constant, longer term rates will be higher than shorter term rates. Stated in terms of forward rates and expected spot rates,

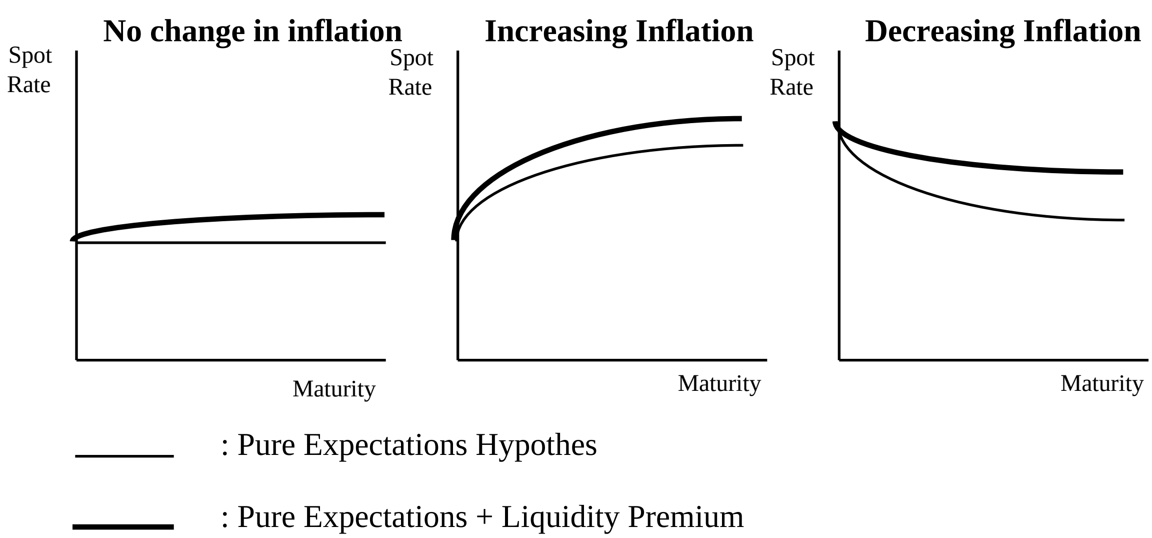
$$_{n-1}F_n = \mathrm{Exp}\left({}_{n-1}S_n\right) + L_t$$

where,

L_t = Liquidity premium corresponding to a bond maturity of t periods

Figure 33.7 illustrates how the liquidity premium builds on top of the pure expectations hypothesis.

Figure 33.7: Term Structure with Liquidity Premium



While the traditional theory assumes a positive liquidity premium (L_t), the assumption that all lenders prefer to lend short term over long term may not be always appropriate. For instance, a lender with fixed liabilities twenty years from now may view a twenty-year zero-coupon bond as less risky than a treasury bill of six months, because it matches cash inflows to cash outflows. The question therefore becomes an empirical one – Does the average lender prefer to lend short or long term?

McCulloch (1975) attempted to estimate term premia for different time periods, and arrived at the following estimates.

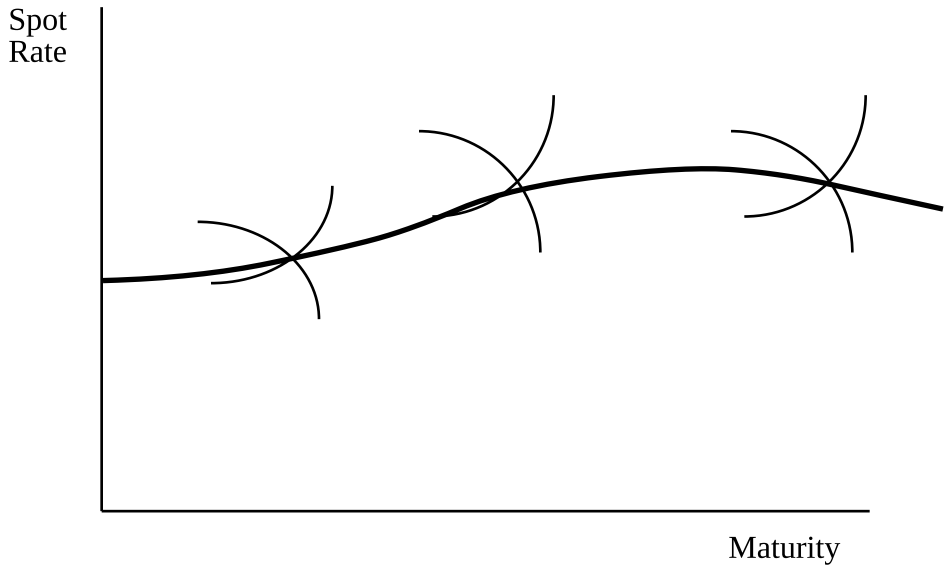
Maturity	*6 month*	*1 year*	*5 year*	*10-year*	*20-year*	*30-year*
Estimate	0.41%	0.43%	0.43%	0.43%	0.43%	0.43%
Standard Error	0.06%	0.07%	0.07%	0.07%	0.07%	0.07%

There are two key findings that emerge from this study. The positive term premia suggest that, on average at least, lenders prefer lending short to long term. The term premia also do not seem sensitive to bond maturity. The second result has been challenged in a number of studies. Van Horne (1965) finds term premia increasing, albeit at a decreasing rate, with bond maturity.

3. Demands from Specific Market Segments

The price of bonds, like any other security, is determined by demand and supply. If the market is segmented, and there are sizable groups of investors whose demand is for a specific maturity, the term structure will be affected by these groups. Again, considering the extreme case, where investors will lend and borrow only for specific maturities, the interest rate at each maturity will be determined by demand and supply at that maturity. This is illustrated in Figure 33.8.

Figure 33.8: Market Segmentation and Term Structure



Under this scenario, the term structure can take any shape, depending upon the demand and supply at each maturity.

The assumption that investors will lend or borrow only for specific maturities and not substitute other maturities even when it is extremely favorable for them to do so is an extreme one. In reality, market segments do exist and do affect the term structure but only at the margin and for one or two maturities. For instance, the demand from Japanese investors in the late eighties for the just-issued thirty year bonds resulted in a slight kink in the term structure, where the thirty-year bond rates were slightly lower than twenty-nine year bond rates, even though the rest of the yield curve was upward sloping.

The Empirical Evidence on Maturity Premia

Empirical studies of the term structure have examined several questions including the relative frequency of upward and downward sloping term structures, the magnitude of liquidity premia and the presence of market segments. The evidence can be summarized as follows.

- The yield curve, at least in this century, has been more likely to be upward sloping than downward sloping. Examining yield curves at the beginning of each year from 1900 to 2000, the yield curve has been downward sloping in only 29 of the 100 years.

This is inconsistent[2] with a pure expectations hypothesis, where downward sloping yield curves should be just as likely as flat or upward sloping yield curves. It is, however, consistent with a combination of the expectation and liquidity preference hypotheses, where positive liquidity premia are demanded over and above expected inflation.

- The term structure is much more likely to be downward sloping when the level of interest rates is high, relative to historical rates. The table below[3] summarizes the frequency of downward-sloping yield curves as a function of the level of interest rates.

	1-year Corporate Bond Rate	*Slope of Yield Curve*		
		Positive	*Flat*	*Negative*
	Above 4.40%	0	0	20
1900-70	3.25% - 4.40%	10	10	5
	Below 3.25%	26	0	0
1971-00	Above 8.00%	4	2	3
	Below 8.00%	13	6	0

This evidence is consistent with the expectations and liquidity preference hypotheses, but it is also consistent with a hypothesis that interest rates move within a normal range. When they approach the upper end (lower end) of the normal range, the yield curve is more likely to be downward sloping (upward sloping).

- Studies have generally found that expectations about future interest rates are important in shaping the term structure. Meiselman computed high positive correlations between forecasting errors and changes in various forward rates, and stable term premiums. In contrast, there are many researchers who argue that the volatility in interest rates is much too great to be explained by just expectations about future rates and constant term premia. Shiller (1979) concludes that the greater the volatility in interest rates, the larger the term premium.

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[2] Prior to the abandonment of the Gold Standard in the 1930s, negatively sloped yield curves were just as likely to occur as positively sloped yield curves.
[3] Some of the data table is extracted from Wood (1984).

- Attempts by the government to alter the shape of the yield curve by adjusting the maturity of issues have largely been unsuccessful in the long term. For instance, "Operation Twist" in 1962 was designed to make the yield curve flatter[4] by lowering long term rates and raising short term rates, by issuing short term debt to finance deficits. Though the yield curve did flatten, long term yields did not decline. This can be viewed as evidence of the weakness of the market segmentation hypothesis.
- There is evidence that the shape of the term structure has strong predictive power for future changes in the real economy. Harvey (1991) examined the G-7 countries (Canada, France, Germany, Italy, Japan, U.K., U.S.A.) and concluded that 54% of world economic growth could be explained the term structure.

c. Default Premium

While there is no possibility of default for bond issues made by the United States Treasury, corporate bonds or state/local bonds can default on interest or principal payments. If there is any possibility of default on a bond, there will be a default premium in addition to the maturity premium on the bond. The default premium will increase with the perceived default risk of the bond and is generally also a function of the maturity and terms of the specific bond. We examined this issue in detail in Chapter 7, as part of the discussion of how best to estimate the cost of debt for a firm. Reviewing that discussion, we concluded that:

- The most direct measure of default risk is the default rate which measures defaulted issues as a percentage of the par value of debt outstanding. Hickman investigated the default experience of fixed-income corporate bonds between 1900 and 1943, as a function of the bond rating.

	Ratings					
Size of Issue	I	II	III	IV	V-IX	No Rating
> $ 5 millions	5.9%	6.0%	13.4%	19.1%	42.4%	28.6%
≤ $ 5 millions	10.2%	15.5%	9.9%	25.2%	32.6%	27.0%

[4] A similar, though less formal, attempt was made in 1993 by the Treasury Department to raise short term rates and lower long term rates by issuing more short term bonds and less long term bonds. It was successful at raising short term rates, but long term rates increased concomitantly.

Hickman's study have been extended by several researchers and data availability has made this easier to do. Altman computes default rates for high yield bonds from 1970 to the present, on an annual basis and relates them to bond ratings.

- Default spreads on bonds tend to increase during economic downturns and decrease during economic booms.
- Default spreads are generally larger for longer term bonds than they are for shorter term bonds, for any given level of default risk. There may be specific circumstances, though, where the reverse is true. Johnson defines a "crisis-at-maturity" scenario, usually in the midst of a recession or a depression, where a firm is perceived to have insufficient funds to meet its immediate debt servicing needs, though it is expected to revert to health in the long term. In this scenario, the default premia will be lower for longer maturity bonds than for shorter maturity bonds. Johnson found evidence of inverted default premia term structures during 1934, in the midst of the depression.

Corporate Bonds in Emerging Markets

In the framework that we have developed, you build up to the rate on a corporate bond by adding a default spread to the government bond rate. This process works only when the government is viewed as having no default risk. When governments have default risk, as is often the case in emerging markets, the process becomes more complicated. To estimate the appropriate interest rate on a corporate bond in an emerging market, you have to begin by estimating a riskless rate. The best way to do it is to build it up from the Fisher equation – add an expected inflation rate to the real rate of return in that market. The latter can be set equal to the expected real growth rate in the economy, but the former can be a volatile number in high inflation markets. An alternative approach is to begin with the government bond rate and subtract out the estimated default spread for the government – this default spread can be obtained using the rating for the government.

You could alternatively estimate the corporate bond rate for a company in an emerging market in a different currency – U.S. dollars or Euros. In this case, the riskless rate will be defined in that currency – the treasury bond rate in the U.S. for dollars and the

German Euro-denominated government bond rate. The default spread for the company can then be added on to this riskless rate to estimate the corporate bond rate.

There is one final point that needs to be confronted with corporate bonds in emerging markets and it relates to whether you should incorporate the country default risk spread into the corporate bond rate. For instance, should the interest rate on a bond issued by Embraer, the Brazilian aerospace firm, incorporate the default spread on Brazilian government bonds? For smaller firms, the answer should generally be yes. For larger firms with substantial operations outside the country, we have a little more leeway. These firms may be able to borrow at rates lower than the sovereign rate.

Special Feature in Bonds and Pricing Effects

In the last section, we examined the question of how to price a government or a corporate bond based upon the expected coupons and the appropriate interest rate for the bond. Most bonds though have other features added on, some of which make the bonds more valuable and some less valuable. In this section, we consider how best to value these special features.

I. Convertibility

A convertible bond is a bond that can be converted into a pre-determined number of shares, at the option of the bondholder. While it generally does not pay to convert at the time of the bond issue, conversion becomes a more attractive option as stock prices increase. Firms generally add conversions options to bonds to lower the interest rate paid on the bonds.

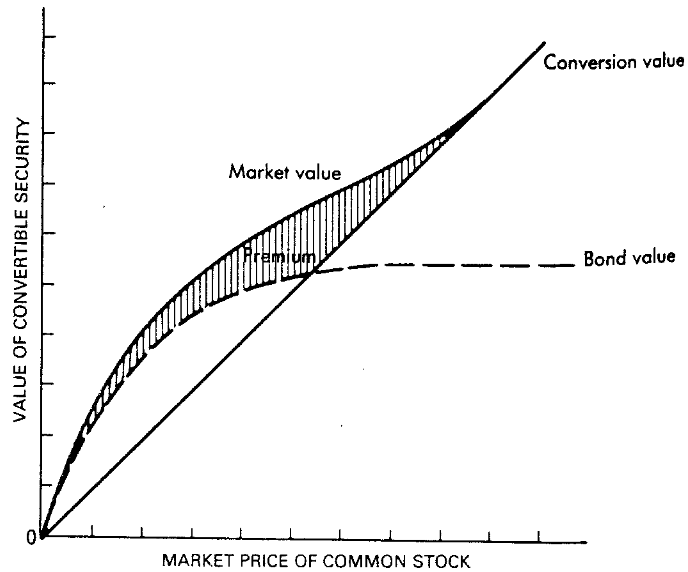
The Conversion Option

In a typical convertible bond, the bondholder is given the option to convert the bond into a specified number of shares of stock. The conversion ratio measures the number of shares of stock for which each bond may be exchanged. Stated differently, the market conversion value is the current value of the shares for which the bonds can be exchanged. The conversion premium is the excess of the bond value over the conversion value of the bond.

Thus a convertible bond with a par value of $1,000, which is convertible into 50 shares of stock, has a conversion ratio of 50. The conversion ratio can also be used to compute a conversion price - the par value divided by the conversion ratio, yielding a conversion price of $20. If the current stock price is $25, the market conversion value is $1,250 (50 * $25). If the convertible bond is trading at $1,300, the conversion premium is $50.

The effect of including a conversion option in a bond is illustrated in Figure 33.9.

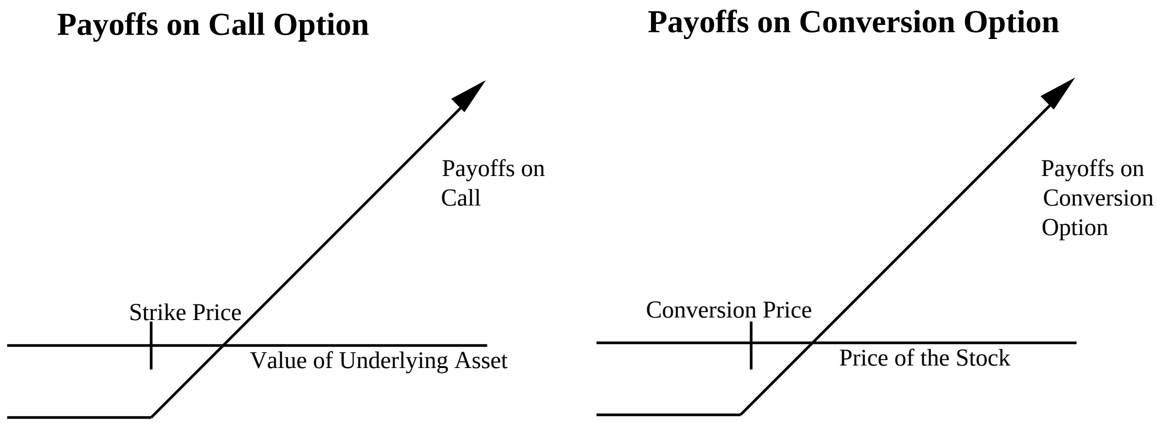
Figure 33.9: Bond Value and Conversion Option



Determinants of Value

The conversion option is a call option on the underlying stock and its value is therefore determined by the variables that affect call option values – the underlying stock price, the conversion ratio (which determines the strike price), the life of the convertible bond, the variance in the stock price and the level of interest rates. The payoff diagrams on a call option and on the conversion option in a convertible bond are illustrated in Figure 33.10.

Figure 33.10: Call Option and Conversion Option: Comparing Payoffs



Like a call option, the value of the conversion option will increase with the price of the underlying stock, the variance of the stock and the life of the conversion option and decrease with the exercise price (determined by the conversion option).

The effects of increased risk in the firm can cut both ways in a convertible bond - it will decrease the value of the straight bond portion while increasing the value of the conversion option. These offsetting effects will generally mean that convertible bonds will be less exposed to changes in the firm's risk than are other types of securities.

Option pricing models can be used to value the conversion option with three caveats – conversion options are *long term*, making the assumptions about constant variance and constant dividend yields much shakier, conversion options result in *stock dilution*, and conversion options are often *exercised before expiration*, making it dangerous to use European option pricing models. These problems can be partially alleviated by using a binomial option pricing model, allowing for shifts in variance and early exercise and factoring in the dilution effect. These changes are described in more detail in Chapter 15. The following illustration provides an example of the use of option pricing models in valuing a conversion option in a convertible bond.

The value of a convertible bond is also affected by a feature shared by most convertible bonds that allow for the adjustment of the conversion ratio (and price) if the firm issues new stock below the conversion price or has a stock split or dividend. In some cases, the conversion price has to be lowered to the price at which new stock is issued. This is designed to protect the convertible bondholder from misappropriation by the firm.

Illustration 33.6: Valuing a conversion option / convertible bond

In December 1994, General Signal had convertible bonds outstanding with the following features.

- The bonds matured in June 2002. There were 100,000 shares of convertible bonds outstanding.
- They had a face value of $1000, and were convertible into 25.32 shares per bond until June 2002.
- The coupon rate on the bond was set at 5.75%.
- The company was rated A-. Straight bonds of similar rating and similar maturity were yielding 9.00%.
- The stock price in December 1994 was $32.50. The volatility (standard deviation in ln stock prices) based upon historical data was 50.00%.
- There were 47.35 million shares of equity outstanding. Exercising the convertible bonds will create 2.532 million additional shares (100,000 * 25.32 shares).

The two components of the convertible bond can be valued as follows.

A. Straight Bond Component

If this bond had been a straight bond, with a coupon rate of 5.75% and a yield to maturity of 9.00% (based upon the bond rating), the value of this straight bond can be calculated.

$$\text{PV of Bond} = \sum_{t=1}^{t=7.5} \frac{28.75}{(1.09)^t} + \frac{1{,}000}{(1.09)^{7.5}} = \$834.79$$

This is based upon semi-annual coupon payments (of $28.75 for semi-annual periods).

B. Valuing the Conversion Option

The value of the conversion option is estimated using the Black-Scholes model, with the following parameters for the conversion option.

Type of Option = Call

Number of Calls/Bond = 25.32

Stock Price = $32.50

Strike Price = $1000/25.32 = $39.49

Time to Expiration = 7.5 years

Standard Deviation in Stock Prices (ln) = 0.5

Riskless rate = 7.75% (Rate on 7.5 year Treasury Bond)

Dividend yield on Stock = 3.00%

Allow for the dilution inherent in the exercise (See chapter 5 on warrant pricing for details on the valuation correction).

Value of one Call = $ 12.85

Value of the Conversion Option = $ 12.85 * 25.32 = $325.43

C. Value of Convertible Bond

The value of the convertible bond is the sum of the straight bond and conversion option components.

Value of Convertible Bond = Value of Straight Bond + Value of Conversion Option

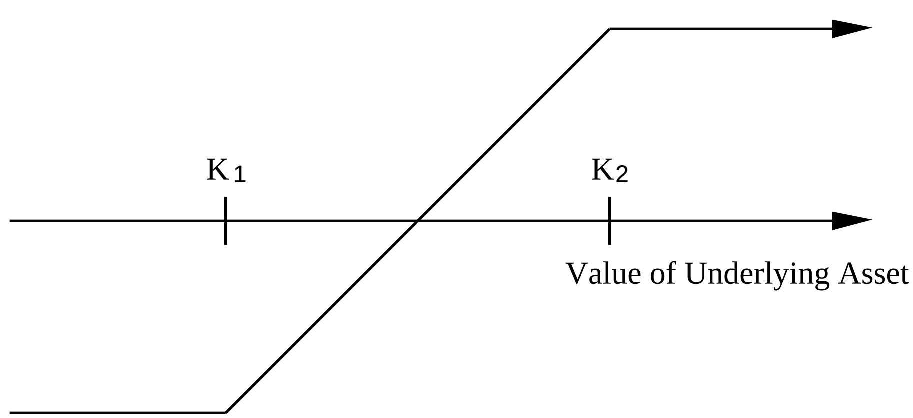
= $ 832.73 + $325.43 = $1158.16

This valuation is based upon the assumption that the conversion option is unconstrained and that the bonds are not callable. The effects of introducing these changes into the analysis will be examined in the following sections.

The Effect of Forced Conversion

Companies that issue convertible bonds sometimes have the right to force conversion if the stock price rises to a specified level. This right to force conversion caps the profit that can be made on the conversion option, and hence affects its value. Figure 33.11 illustrates the effect of forced conversion on the expected payoffs.

Figure 33.11: Value of a Capped Call



The value of a capped call, with an exercise price of K_1 and a cap of K_2 can be calculated as follows.

Value of capped call (K_1, K_2) = Value of Call (K_1) - Value of Call (K_2)

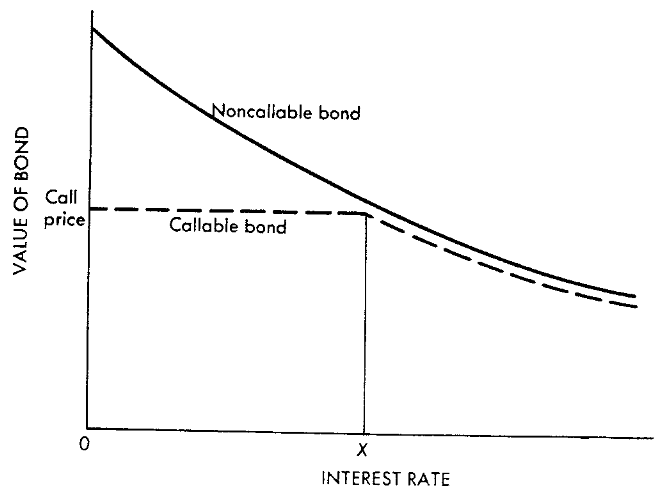
This is because the cash flows on a capped call can be replicated by buying the call with a strike price of K_1 and selling the call with a strike price of K_2.

II. Callability

The issuer of a callable bond preserves the right to call back the bond and pay a fixed price (generally at a premium over the par value) for it. Thus, if interest rates decline (bond prices rise) after the initial issue, the firm can refund the bonds at the fixed price instead of the market value. Adding the call option to a bond should make it less attractive to buyers, since it reduces the potential upside on the bond. As interest rates go down, and the bond price increases, the bonds are more likely to be called back.

The distinction between a straight bond and a callable bond are illustrated in the Figure 33.12.

Figure 33.12: Callable versus Straight Bonds



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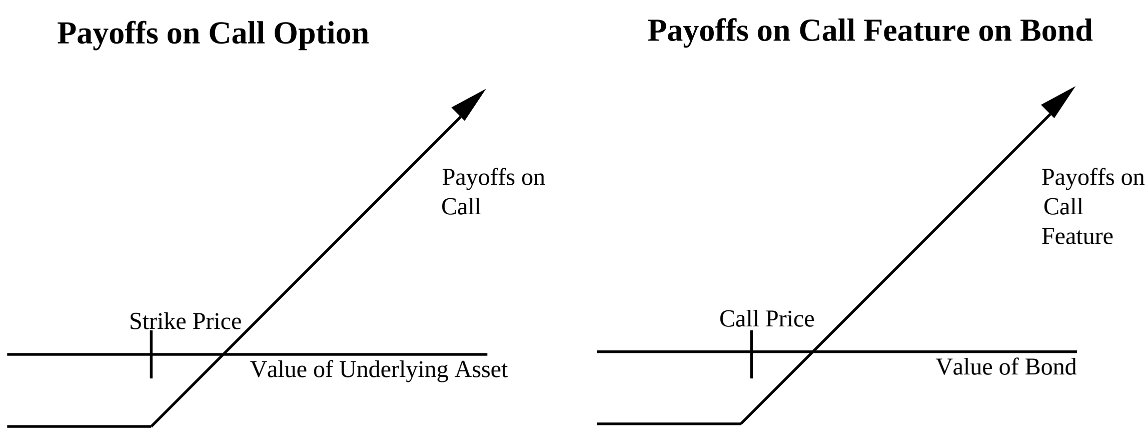
The difference on the upside between straight and callable bonds is quite clearly illustrated in Figure 33.12. As interest rates decline, the values of the two bonds diverge, whereas they converge as interest rates increase.

There are several common features shared by most callable bonds. Most callable bonds come with an initial period of *call protection*, during which the bonds cannot be called back. Such bonds are called deferred callable bonds. The call price on most callable bonds is set at an initial level above par value plus one annual coupon payment, but declines as time passes and approaches the par value.

Valuing the Callability Option

The issuer's right to call back a bond if interest rates drop (or bond prices rise) to an attractive level is a call option on the bond and can be valued as such. The payoffs on a callable bond are shown in Figure 33.13.

Figure 33.13: Payoffs on Call Feature on Bond to Seller of Bond



The value of the callable feature on a callable bond will increase as interest rates decline, and as the volatility of interest rates increases. Since the callable feature is held by the issuer of the bond, the value of a callable bond can be written as follows:

Value of Callable Bond = Value of Straight Bond - Value of Call Feature in Bond

A callable bond should therefore sell for less than an otherwise similar straight bond.

Traditional Analysis

The traditional approach to analyzing callable bonds is to estimate **yields to call** as well as yields to maturity. The former is based upon the assumption that the bond will be called at the first call date while the latter assumes holding the bond until maturity. The two yields are compared and the investor chooses the lower of the two as a measure of his expected return on the bond. This approach can also be extended to calculate the yield to all possible call dates and picking the lowest of these yields as the expected yield on the callable bond. This yield is called the **yield to worst**.

While this approach may give the investor some sense of the potential downside from the callability of the bond, it suffers from all the standard problems of the 'yield to maturity' calculation. First, it assumes that the investor can reinvest all coupons until the bond is called at the yield to call, which is not a realistic assumption since calls are much more likely if interest rates go down. Second, it does not examine the rate at which the proceeds from the called bond can be reinvested by the investor. Third, it assumes that the bond will be called on the call date, which takes away the option characteristics of the call feature.

Illustration 33.7: Estimating yields to maturity and call on a callable bond

Consider a corporate bond, with 20 years to maturity and a 12% coupon rate that is callable in two years at 105% of the face value. The bond is trading at 98 currently. The yields to maturity and the yields to call on the corporate bond are as follows:

$$\text{Price} = \sum_{t=0.5}^{t=20} \frac{60.00}{(1+r)^t} + \frac{1,000}{(1+r)^{20}} = \$980$$

The yield to maturity, r, is approximately 12.26%.

The yield to call can be similarly calculated.

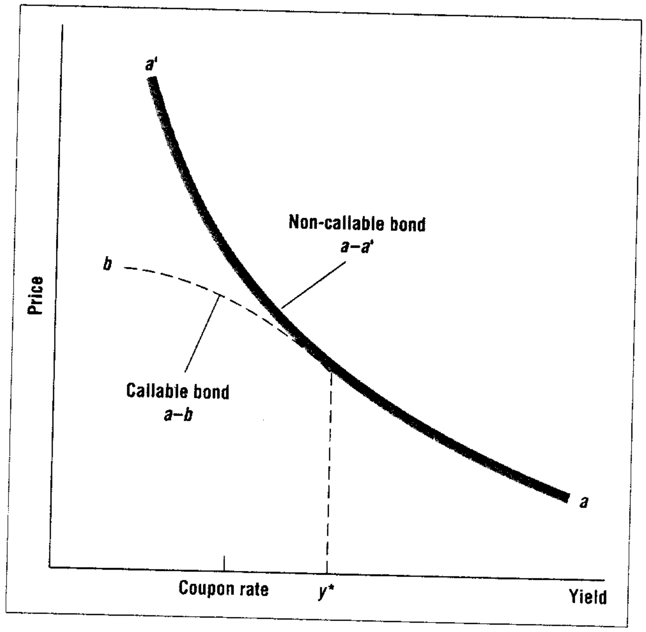
$$\text{Price} = \sum_{t=0.5}^{t=2} \frac{60.00}{(1+r)^t} + \frac{1,000}{(1+r)^{2}} = \$1050$$

The yield to call is approximately 13.25%.

Price/Yield Relationship for a Callable Bond

The price/yield relationship on a callable bond is different because the potential that the bond will be called back puts an upper limit on the price. This makes the relationship between price and yield convex, for some range of the yields. The difference is illustrated in Figure 33.14.

Figure 33.14: Callable Bond Prices and Interest Rates



The section of the price/yield relationship on the callable bond when the yield falls below y* has negative convexity - i.e., the price appreciation on this bond will be less than the price depreciation for a given change (down or up) in interest rates.

Determinants of Value - Option Pricing Approach

The call feature in a callable bond can be valued using option pricing models. It is a series of call options on the underlying bond and its value is determined by the level and volatility of interest rates. There are some modifications that need to be made to the standard option pricing models before they can be applied in this context.

Once the call feature is valued as a series of option, the yield on a callable bond can be adjusted for the option features and the difference between this adjusted yield and treasuries of equivalent maturity is called the **option adjusted spread**. This approach is a more realistic way of considering the effects of the call feature on expected yields than the traditional yield to call approach.

The following illustration values the call feature on a callable bond.

Illustration 33.8: Valuing a callable bond

The following analysis values a 17-year callable bond with a coupon rate of 12% by valuing the straight bond, the call feature on the straight bond and the value of the callable bond as a function of the yield on the bond. The actual option valuation was done using a binomial option pricing model, using an interest rate volatility of 12% and a short term interest rate of 6%.

Yield	*Value of Straight Bond*	*Value of Call Feature*	*Value of Callable Bond*
20.51%	$ 60.00	$ 0.00	$ 60.00
19.55%	$ 63.00	$ 0.00	$ 63.00
18.66%	$ 66.00	$ 0.00	$ 66.00
17.59%	$ 70.00	$ 0.00	$ 70.00
16.63%	$ 74.00	$ 0.00	$ 74.00
15.54%	$ 79.00	$ 0.02	$ 78.98
14.56%	$ 84.00	$ 0.06	$ 83.94
13.51%	$ 90.00	$ 0.22	$ 89.78
12.57%	$ 96.00	$ 0.67	$ 95.33
11.46%	$104.00	$ 2.11	$101.89
10.59%	$111.00	$ 4.60	$106.40
9.59%	$120.00	$ 9.80	$110.20
8.60%	$130.00	$17.81	$112.19
7.73%	$140.00	$27.21	$112.79

While the value of the straight bond increases as the yield drops, the callable bond’s value stops increasing because the call feature becomes more and more valuable as the yield becomes lower. In fact the value of the callable bond is maximized at $112.94.

Effective Duration and Effective Convexity

In the previous section we defined duration to be a measure of a bond’s sensitivity to interest rate changes. While doing so, it was assumed that cash flows did not change as interest rates changed. This assumption is clearly violated for callable bonds, where the cash flows on the bond are influenced by the level of rates - if interest rates drop enough, the bond will be called. For bonds such as these, there is a different measure of duration that is more appropriate called the *effective duration*. The duration of any bond can be approximated as follows, for a small change in interest rates.

$$\text{Duration} = \frac{P_{-} - P_{+}}{P_0 \left(y_{+} - y_{-} \right)}$$

where P_{-} = Price of the bond if yield drop by x basis points

P_{+} = Price of the bond if yield increases by x basis points

P_0 = Price of the bond initially

y_{+} = Initial yield + x basis points

y_{-} = Initial yield - x basis points

This approach can be used to estimate the effective duration of callable bonds for any segment of the yield curve. It can also be used for any other bonds with embedded options, such as putable bonds, or mortgage backed securities, which have the prepayment option embedded in them.

A similar adjustment can be made to the standard convexity measure to arrive at the *effective convexity* of any bond with embedded options. [NOTE: There was no equation for convexity.]

$$\text{Effective Convexity} = \frac{P_{+} + P_{-} - 2P_0}{P_0 \left(0.5 \left(y_{+} - y_{-} \right) \right)^2}$$

Valuing a Callable-Convertible Bond

Many convertible bonds have embedded call features. The presence of two options in the bond, one possessed by the buyer of the bond and the other possessed by the seller of the bond, and the interaction between the two options, implies that the two options have to be valued together. Brennan and Schwartz (1977, 1980) provide an analysis of convertible bonds with call features, default risk and stock price dilution. The simplest approach for illustrating the interaction between the various options is a binomial option pricing model.

Empirical Evidence on Call Feature

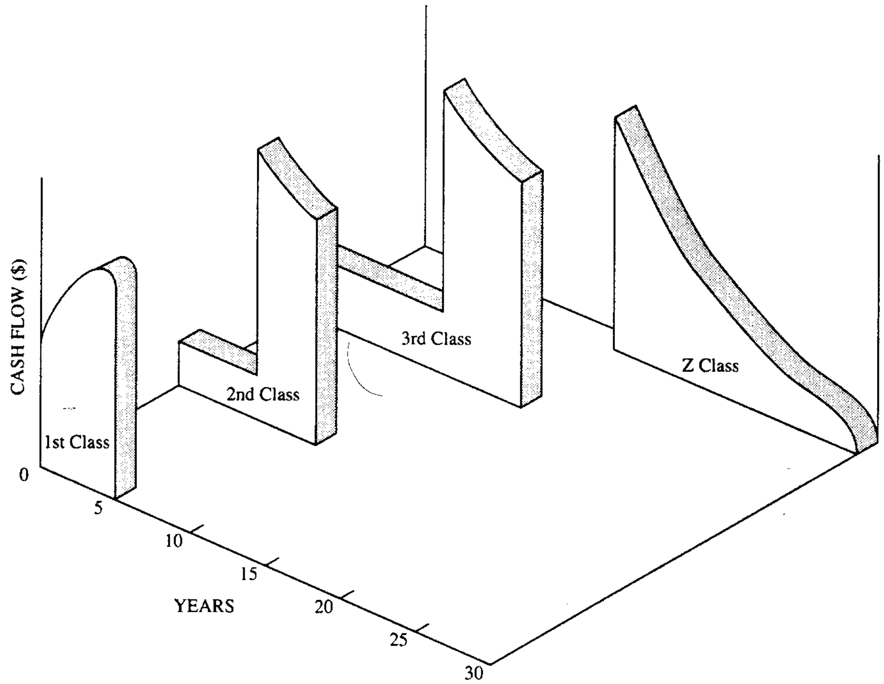
When a convertible bond is callable, holders of the convertible bond lose the opportunity to make further returns on the bond as stock prices increase. Companies can establish a variety of call policies such as calling the instant the market value of the convertible rises above the call price or waiting until the market value is well in excess of the call price. Ingersoll (1977) argues that a bond should be called when its conversion value equals its call price. Given that a thirty-day notice has to be given to bondholders of a call, firms may prefer to build a cushion to protect against risk during this period.

The empirical evidence however suggests that firms do not usually follow the optimal policy. Ingersoll, for instance, finds that between 1968 and 1975, the average conversion value was 43.9% above the call price for bonds and 38.5% for preferred stocks. The call policy chosen by a firm and communicated to financial markets implicitly through its actions, has an effect on the value of the convertible bond.

III. Pre-payment Option

Mortgage backed securities, which came of age in the eighties, securitized residential mortgages, by packaging them and issuing marketable securities of various types on them – either as flow through investments where holders receive a share of the total cash flows on the pool of mortgages or as derivative products, where holders receive customized packages of cash flows depending upon their preferences. The latter, called collateralized mortgage obligations, in its simplest form, divide cash flows on the mortgage pool into four tranches, with cash flows on each tranche starting as the cash flows on the prior tranche are completed. Figure 33.15 illustrates this type of security.

Figure 33.15: Cash flows on a Mortgage Pool

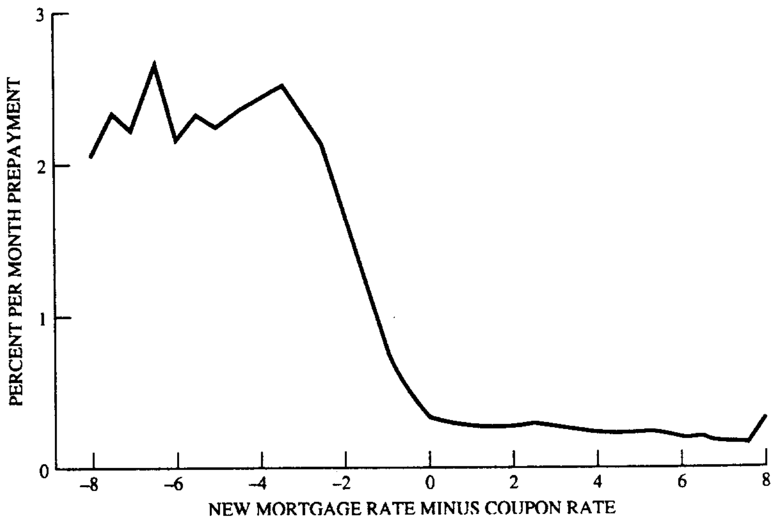


In recent years, CMOs have been refined further and even more specialized products have been created including stripped mortgage-backed securities (where cash flows are divided on the basis of principal and interest), floating rate classes and inverse floaters (where the interest rate on the security increases as the index rate decreases).

Mortgages can be pre-paid by borrowers, if interest rates decline. This prepayment option that resides with borrowers affects the cash flows, and therefore the value, of all mortgage-backed securities.

The Prepayment Option

The homeowner may prepay a loan for any number of reasons, but the level of interest rates is a critical variable. If interest rates declines sufficiently, the potential gain from pre-payment may exceed the cost of pre-payment. The following graph illustrates the percentage of homeowners who prepay as a function of the difference between interest rate and the coupon rate, based upon historical data.



Source: Sean Becketti, "The Prepayment Risk of Mortgage-backed Securities," *Economic Review of the Federal Reserve Bank of Kansas City* (February 1989), 53.

If the level of interest rates were the only determinant of prepayment and homeowners were rational about prepayment decisions, the prepayment option could be valued very similarly to the call option in a callable bond (as a function of the level and volatility of interest rates).

There are, however, other variables besides the level of interest rates that determine whether homeowners prepay. For instance, there is a correlation between prepayment and the age of a mortgage, irrespective of interest rates. Furthermore, some homeowners may never prepay their mortgages no matter how much interest rates drop. There are also seasonal factors that affect prepayment. Consequently, option pricing models alone fall short in pricing prepayment options in mortgage backed securities.

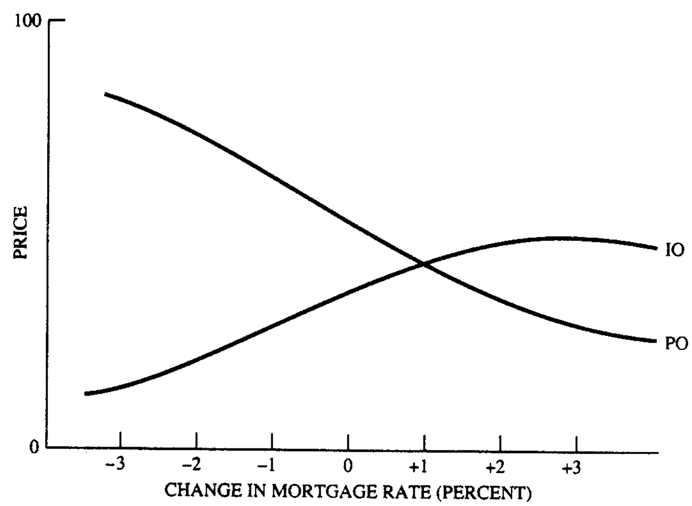
A number of researchers have attempted to develop models that explain prepayment, as a basis for pricing the prepayment option, with characteristics such as age and coupon rate as inputs, in addition to specific characteristics of the borrowers in the pool. In cases where a specific rather than a generic pool of mortgages is being priced, the historical payment record of the specific[5] pool is useful and is often the basis for estimating prepayments.

[5] A number of variables have been found to be useful in explaining prepayments - the market price relative to the original purchase price and geographical differences, for instance.

Valuing the prepayment option

The effect of the prepayment option on value will vary with the type of mortgage backed security. Consider, for instance, the price behavior of interest-only and principal-only securities, as interest rates changes. As interest rates increase, the interest payments on the interest-only securities goes up, leading to a higher value for the security, at least initially, though the present value effects (which are negative) start to dominate beyond a certain point. As interest rates decrease, the prepayments lead to lower interest payments and a lower value for the security. The principal-only securities behave more like conventional bonds, increasing in value as interest rates decline and decreasing in value as they increase. Figure 33.16 illustrates this relationship.

Figure 33.16: Mortgage Rates and Security Values



***IO**: Interest Only Security* ***PO**: Principal Only Security*

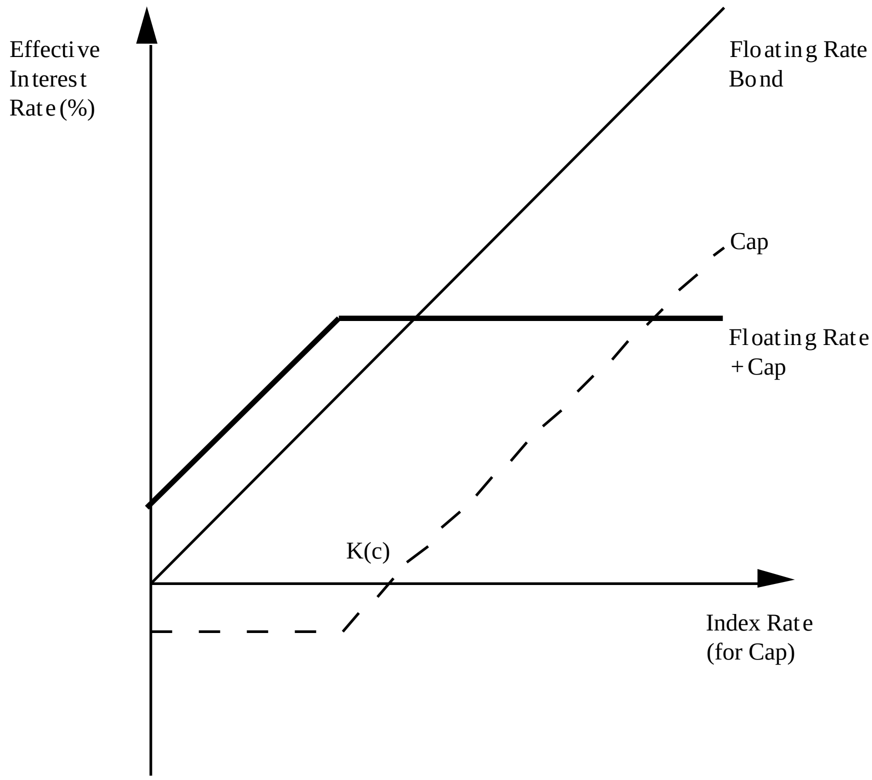
IV. Interest Rate Caps and Floors

A floating rate bond is a bond which has an interest rate linked up to an index - either a government bond rate (treasury bond or bill) or to the LIBOR. The rationale for issuing such bonds is to reduce the interest rate risk for both the issuer and the buyer of the bond. Most floating rate bond issuers, however, **cap** their floating rate obligations to ensure that interest rates do not rise above a pre-specified rate (the cap). Some floating rate bonds offer buyers some compensation by providing a **floor**, below which interest rates will not decline. If a floating rate bond has a cap and a floor, a **collar** is created.

Caps, Floors and Collars

The presence of a cap on a floating rate bond can be illustrated best by contrasting a bond with a cap against a floating rate bond without one, as shown in Figure 33.17.

Figure 33.17: Effects of Caps on Floating Rate Loans



The cap on a floating rate bond has the same effect as a call option on interest rates with a strike price of K_c, with the issuer of the bond holding the option. A call option on interest rates translates[6] into a put option on the underlying bond. The price of a floating rate bond with a cap can then be written as:

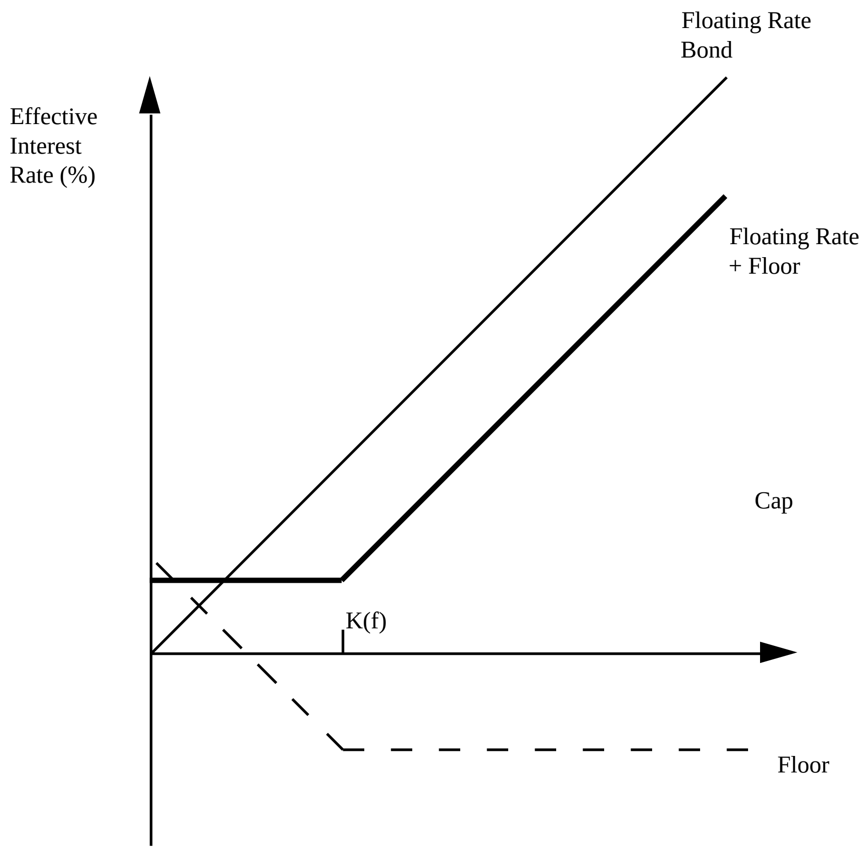
Price of floating rate bond with cap = Price of floating rate bond without cap

- Value of put on bond

The presence of a floor on interest rates can also be illustrated using a similar comparison of a bond with a floor against a bond without one in Figure 33.18.

[6] The translation is not one to one. A call option on interest rates is the equivalent of α options on the underlying bill or bond, where α = 1/ Exercise price of equivalent bill.

Figure 33.18: Effects of Caps on Floating Rate Loans

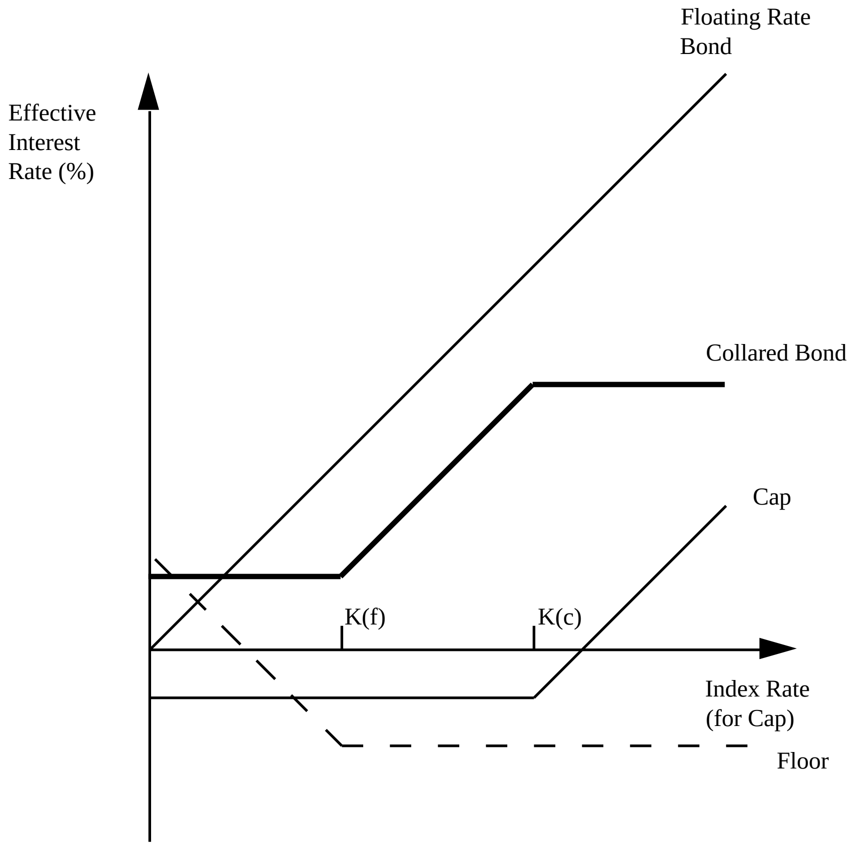


The floor on a floating rate bond has the same effect as adding a put option on interest rates with a strike price of K_f, with the buyer of the bond holding the put. A put option on interest rates can be translated into a call option on the underlying bond. The price of a floating rate bond with a floor can then be written as:

Price of floating rate bond with floor = Price of floating rate bond without cap
+ Value of call on bond

Finally, the presence of both a cap and a floor can be illustrated in Figure 33.19.

Figure 33.19: Effects of Caps on Floating Rate Loans



The presence of a collar on a floating rate bond creates two options - a call option with a strike price of K_c for the issuer of the bond and a put option with a strike price of K_f for the buyer of the bond. These options on interest rates can be stated again in terms on options on the underlying bond.

Price of floating rate bond with collar = Price of floating rate bond without collar

+ Value of call on bond

- Value of put on bond

Valuing caps and floors

Option pricing models can be used to value caps, floors and collars with some caveats. The key assumption in the Black Scholes model of constant volatility over the life of the option is likely to be violated for interest rate options, both because of the long term nature of these options and because the variance in the bond price is likely to change as the bond approaches maturity. There have been attempts to use yield instead of price and assume that it conforms to a lognormal distribution.

Subrahmanyam (1990) notes that the value of a cap on interest rates can be written as a series of put options on the price of an equivalent bill or bond. Briys, Crouhy and Schobel (1991) provide a framework for pricing caps, floors and collars. They argue that caps and floors can be modeled as a series of independent options on zero coupon bonds. They allow for the fact that bond prices do not follow the geometric Brownian motion used by Black and Scholes (1973), but adopt a different stochastic process to price caps, floors and collars.

Illustration 33.9: Valuing a 2-year Cap/Floor on 6 Month LIBOR

Assume that the current 6-month LIBOR rate is 8%, and that the cap/floor has an exercisable price of 8%. The cap consists of three options which are exercisable at the end of 6 months (183 days), 12 months (365 days) and 18 months (548 days). Each of these options is on the $LIBOR rate (of 183 days for the first, 182 days for the second and 183 days for the third).

The options can be valued using the bill prices (rather than interest rates) and the inputs used in the Black-Scholes Model are as follows.

Option Maturity	Bill price	Exercise Price	Forward Price	Volatility
183	0.9609	0.9611	0.9626	0.0100
365	0.9250	0.9609	0.9609	0.0100
548	0.8890	0.9611	0.9615	0.0100

The first column provides the maturity period for each of the three options. The second column is the value of a zero-coupon bond with a maturity equal to the maturity of the zero-coupon bond - $1 discounted back 183 days at 8% is $0.9609, $1 discounted back 365 days at 8% is $0.9250 and so on. The third column is the strike price, based upon the cap rate, for each option. The fourth column is the forward price of the bill at the option expiration date. The final column is the assumed annualized volatility in bill prices of 1%.

The values of the call and puts options with these maturities on bills is provided in the following table in basis points. These can be converted into option values for options on interest rates by multiplying by the number of put options on bills that is equivalent to one call option on the interest rate.

Adjustment Factor = α = 1/Exercise price of equivalent bill option

	T. Bill Option		*Interest Rate Options*	
	Put	*Call*	*Call*	*Put*
183	19.6133	33.6639	20.4065	35.0254
365	34.6903	36.4439	36.1010	37.9259
548	39.8620	43.5135	41.4742	45.2734
Value of the Cap =				**97.9817**
Value of the Floor =				**120.2237**

To illustrate the calculation, the values of the interest rate options for the 183 day option can be estimated as follows.

Adjustment factor for 183 day option $= \frac{1}{0.9611} = 1.0405$

Value of 183-day put on T.Bill = 19.6133

Value of 183-day call on interest rate = 19.6133 * 1.0405 = 20.4065

Valuing Options Embedded in Bonds

A corporate bond can often have three or four options embedded in it and to value the bonds, you have to value the options. While conventional option pricing models can be used to value fixed income options, you should note the following.

- The assumption of constant volatility that we often use to value options on stocks cannot be used to value options on bonds such as callability. Bonds are finite life instruments and their volatility will decrease as they approach maturity. You will have to model the change in volatility over time to price the option.
- When multiple options exist in a bond, you will have to examine the relationship between the options to price them. For instance, consider a callable, convertible bond. While both callability and convertibility are options – one is held by the bond issuer and the other by the bond buyer – the exercise of one of these options voids the other. This will become a factor when the options will be exercised and how much they are worth.
- The key underlying variable for some bond options – such as interest rate caps and floors – is the interest rate process and how it is modeled can have a significant impact on the value of the options.

V. Other Features

There are a number of other bond features which affect the value of the bond - a sinking fund provision, where the firm plans to retire a specified face value of the bonds outstanding each year, provisions relating to the subordination of future debt issues and bond covenants on investment and dividend policy.

Sinking Funds

Most industrial bond issues come with sinking fund provisions, requiring the issuer to retire a specified portion of the bond issue each year, starting a period of time (five or ten years) after the initial issue. The sinking fund provision can take one of two forms.

(a) A trustee collects a cash payment from the bond issuer and calls bonds for redemption at the sinking-fund call price, usually based upon a lottery.

(b) The bond issuer can buy back bonds in the open market and deliver the specified number of bonds to the trustee in the periods specified.

If the bond issuer has the option to do the latter, bonds will be bought back and delivered if the market price is less than the call price and cash will be delivered to the trustee to make the call if the market price is greater than the call price.

Sinking funds usually relate to a single issue, but they can sometimes cover multiple issues ("funnel sinking fund"). Most sinking funds also allow the bond issuer to accelerate call backs if it is in the issuer's favor to do so (i.e., interest rates have gone down since the issue).

A sinking fund has two effects, one of which benefits the issuer of the bond and the other which benefits the buyer of the bond. The issuer of the bond gets a delivery option, because he has an option to either deliver the cash for the call price or to buy the bonds at the market price. The value of this call option (similar to the option in a callable bond) will increase with the volatility of interest rates and decrease with the level of interest rates. The buyer of the bond has less default risk because of the requirement that some of the debt be retired each period. The net effect will determine whether a sinking fund provision adds or detracts from the value of a bond.

The empirical evidence on the sinking fund provision is mixed. While some of the earlier studies concluded that a sinking fund provision added to bond value, Ho and Lee (1985) find that its net value is insignificant overall, but that it adds more value as default risk increases than it does as interest rate volatility increases.

Subordination of Further Debt and Collateral

Existing debt holders are negatively affected by the issue of new debt, especially if the new debt has superior claims on the assets of the issuer. Therefore, some bond issues have *subordination clauses,* which put restrictions on the issue of additional debt. Additional debt might have to be subordinated to existing debt; i.e., in the event of bankruptcy, subordinated debt will be paid off after existing debt is fully paid. The presence of subordination clauses in a bond agreement should make it less risky and therefore more valuable.

Some bonds are issued with specific collateral issued behind them, with a specific asset of the firm backing up the promised payments on the bond. If the collateral is property, the bond is called a mortgage bond, whereas, if it is securities, it is a collateral trust bond. Other bonds are issued without specific collateral and are called unsecured bonds. Other things remaining equal, secured bonds should be viewed as less risky and more valuable than equivalent unsecured bonds.

The Effect of Bond Covenants

Most bond issues are accompanied by a set of covenants that restrict the investment and dividend policies of the firm. These covenants are designed to protect bondholders from stockholders, who might try to expropriate wealth from them by:

(a) investing in much riskier projects, especially if the firm is highly levered, or

(b) paying significantly higher dividends than expected.

Bond covenants should reduce the risk of expropriation on a bond and increase the value of the bond.

Conclusion

The price of a bond is the present value of the cashflows on the bond – coupons and face value – discounted back at an appropriate interest rate. To estimate that interest rate, we began with the instantaneous riskless interest rate and added a maturity premium and a default premium to it.

Bonds become increasingly complex as special features are added to them, since these special features affect the cash flows, risk and value of these bonds. Many of these special features have option characteristics - the chance to convert the bond into other securities or assets, the option to call the bond back if interest rates go down and the option to put the bond back to the issuer if contractual obligations are not met. Traditional option pricing models can be used to value these options, some of which reside with the buyer (thus increasing the value of the bond) and some of which reside with the seller (which would reduce value). The presence of more than one of these options in the same bond (for example, a callable convertible bond) does add to the complexity of the process, but it can be overcome.

Problems

1. Estimate the value of a just-issued 20-year government bond with an 8% coupon rate if interest rates are at 9%. How much will this value change if interest rates go up by 2%? if they go down by 2%? (Coupons are paid semi-annually.)

2. Estimate the value of seasoned government bond with a 7.5% coupon rate and twelve years to maturity, if interest rates are at 8.0%. (Coupons are paid semi-annually, and the next coupon is due in three months.)

3. Estimate the duration of a government bond with a coupon rate of 10% and a 5-year maturity, if the yield to maturity on the bond is 8%. (You can assume, for purposes of simplicity, that the coupons are paid annually.)

4. Why are longer-term bonds more sensitive to a given change in interest rates than shorter term bonds? Why are zero-coupon bonds more sensitive than coupon bonds of equal maturity?

5. If the nominal interest rate is 8% and the expected inflation is 5%, estimate the expected real rate of return. Why might the actual real rate of return deviate from this expectation?

6. You are provided with the following information on government bonds of different maturities.

Maturity	Yield to Maturity
1 year	5.0%
2 years	5.5%
3 years	6.0%
4 years	6.5%
5 years	7.0%

You can assume that the bonds are trading at par and, therefore, the coupon rates are equal to the yields to maturity.

   a. Plot the yield curve using the yields to maturity.
   b. Estimate the spot rates for the different maturities.
   c. Estimate the forward rates for each of the four one-year periods.

7. If lenders demand a liquidity premium for lending long term, yield curves will always be upward sloping. Is this statement true? Why or why not?

8. Some studies that looked at low-rated bonds in the 1980s found that the default premiums received on these bonds were much larger than the default rate on them. (In other words, investors in these bonds made more over the period, even after adjusting for actual defaults, than investors in higher-rated or default free bonds.) They then concluded that the default premiums were too high. Would you agree? Why or why not?

9. You are analyzing a convertible bond with a face value of $1000 and an annual coupon of 4%, which is convertible into 30 shares of stock anytime over the next 20 years. The current stock price is $27 and the convertible is trading at $1177. Estimate the following:
   a. the conversion ratio and conversion price.
   b. the conversion premium.
   c. if the interest rate on straight bonds issued by the same company is 8%, estimate the value of the conversion option.

10. ITC Corporation has convertible bonds outstanding with the following features:
    · The bonds mature in fifteen years; there are 100,000 bonds outstanding.
    · Each bond can be converted into 50 shares of stock any time until expiration.
    · The coupon rate on the bond is 5%; straight bonds issued by the company are yielding 10%.
    · The current stock price is $15 per share and the standard deviation in ln(stock prices return) is 40%.
    · There are 20 million shares outstanding.
    a. Value the conversion option.
    b. Estimate the value of the straight bond portion.
    c. If these bonds were issued at par, who would be gaining? Who would be losing?
    d. What impact would forced conversion have on the value of this convertible bond?

11. A company has two issues of bonds outstanding - they both have the same maturities and coupon rates, but differ in one respect. The first issue (Issue A) is callable, while the second is not. Respond true or false to the following statements.
    a. The callable bonds will trade for a higher price than the non-callable bonds.
    b. The callable bonds have a shorter duration than the non-callable bonds.
    c. The callable bonds will have a higher yield than the non-callable bonds.
    d. The callable bonds will be more sensitive to interest rate changes than the non-callable bonds.

12. You are evaluating the yield on a callable bond, with a 10-year maturity and a 9% coupon rate. The bonds can be called back at 110% of par in 3 years. The bond is trading at $950.
    a. Estimate the yield to maturity.
    b. Estimate the yield to call.
    c. Which of the two would you use in analyzing the bond?

13. Collateralized Mortgage Obligations (CMOs) provide investors with the opportunity to invest in cash flows from mortgage obligations. These cash flows are affected by mortgage prepayments. Assume that you have valued (and bought) CMOs on the assumption that homeowners will prepay as soon as it is rational for them to do so. What would be the effect on your returns if:
    i. homeowners consistently waited too long before prepaying mortgages.
    ii. homeowners consistently prepaid mortgages at the right time.

14. Answer true or false to the following statements and explain.
    a. A floating rate loan with no cap or floor has very low or no duration.
    b. A floating rate loan with a cap will have a higher interest rate than a similar floating rate loan with no cap.
    c. A floating rate loan with a floor will have a higher interest rate than a similar floating rate loan with no floor.
    d. A loan with a sinking fund provision will have a lower interest rate than a similar loan with no sinking fund provision.