

CHAPTER 11: THE BLACK-SCHOLES ANALYSIS

11.1 Lognormal Property of Stock Prices

A variable has a lognormal distribution if the natural logarithm of the variable is normally distributed. In Section 10.6 we showed that if a stock price follows geometric Brownian motion,

$$dS = \mu S dt + \sigma S dz$$

Then

$$d \ln S = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dz$$

From this equation we see that the variable $\ln S$ follows a generalized Wiener. The change in $\ln S$ between time t and T is normally distributed:

$$\ln S_T - \ln S \sim \phi \left[\left(\mu - \frac{\sigma^2}{2} \right) (T-t), \sigma \sqrt{T-t} \right] \quad (11.1)$$

From the properties of a normal distribution it follows from this equation that

$$\ln S_T \sim \phi \left[\ln S + \left(\mu - \frac{\sigma^2}{2} \right) (T-t), \sigma \sqrt{T-t} \right] \quad (11.2)$$

This shows that $\ln S_T$ is normally distributed so that S_T has a lognormal distribution. A variable that has a lognormal distribution can take any value between zero and infinity. Unlike the normal distribution, it is skewed so that the mean, median, and mode are all different. From the last equation and the properties of the lognormal distribution, it can be shown that the expected value of S_T , $E(S_T)$, is given by

$$E(S_T) = S e^{\mu(T-t)}$$

The variance can be shown to be given by

$$\text{var}(S_T) = S^2 e^{2\mu(T-t)} \left[e^{\sigma^2(T-t)} - 1 \right]$$

11.2 The Distribution of the Rate of Return

The lognormal property of stock prices can be used to provide information on the probability distribution of the continuously compounded rate of return earned on a stock between times t and T . Define the continuously compounded rate of return per annum realized between t and T as η . It follows that

$$S_T = S e^{\eta(T-t)}$$

and

$$\eta = \frac{1}{T-t} \ln \frac{S_T}{S}$$

Equation (11.1) implies that

$$\ln \frac{S_T}{S} \sim \phi \left[\left(\mu - \frac{\sigma^2}{2} \right) (T-t), \sigma \sqrt{T-t} \right]$$

and it follows that

$$\eta \sim \phi \left(\mu - \frac{\sigma^2}{2}, \frac{\sigma}{\sqrt{T-t}} \right)$$

What Is the Expected Rate of Return?

In Chapter 10, μ was defined as the expected value of the rate of return in any short interval. How can this be different from expected value of the continuously compounded rate of return in a longer time interval?

We should expect that the expected rate of return in a very short period of time to be greater than the expected continuously compounded rate of return over a long period of time. The expected rate of return in an infinitesimally short period of time is μ . The expected continuously compounded rate of return is $\mu - \sigma^2/2$. These arguments show that the term *expected return* is ambiguous. Unless otherwise stated, we will use it to refer to μ throughout this book.

11.3 Estimating Volatility from Historical Data

To estimate the volatility of a stock price empirically, the stock price is usually observed at fixed intervals of time. Define:

$n+1$: number of observations

S_i : stock price at end of i th interval ($i = 0, 1, \dots, n$)

τ : length of time interval in years

And let

$$u_i = \ln \left(\frac{S_i}{S_{i-1}} \right)$$

For $i=1,2,\dots,n$.

Since $S_i = S_{i-1}e^{u_i}$, u_i is the continuously compounded return (not annualized) in the i th interval. The usual estimate, s , of the standard deviation of the u_i 's is given by

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (u_i - \bar{u})^2}$$

or

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^n u_i^2 - \frac{1}{n(n-1)} \left(\sum_{i=1}^n u_i \right)^2}$$

From equation (11.1), the standard deviation of the u_i 's is $\sigma\sqrt{\tau}$. The variable, s , is therefore an estimate of $\sigma\sqrt{\tau}$. It follows that σ itself can be estimated as s^* , where

$$s^* = \frac{s}{\sqrt{\tau}}$$

The standard error of this estimate can be shown to be approximately $s^*/\sqrt{2n}$.

Choosing an appropriate value for n is not easy. *Ceteris paribus*, more data generally lead to more accuracy. However, σ does change over time and data that are too old may not be relevant for predicting the future. A compromise that seems to work reasonably well is to use closing prices from daily data over the most recent 90 to 180 days. A rule of thumb that is often used is to set the time period over which the volatility is measured equal to the time period over which it is to be applied.

This analysis assumes that the stock pays no dividends, but it can be adapted to accommodate dividend-paying stocks. The return, u_i , during a time interval that includes an ex-dividend day is given by

$$u_i = \ln \frac{S_i + D}{S_{i-1}}$$

The return in other time intervals is still

$$u_i = \ln \left(\frac{S_i}{S_{i-1}} \right)$$

11.4 Concepts Underlying the Black-Scholes differential Equation

The Black-Scholes differential equation is an equation that must be satisfied by the price, f , of any derivative dependent on a non-dividend-paying stock. The Black-Scholes analysis is analogous to the no-arbitrage analysis we used in Chapter 9 to value options when stock price changes are binomial. The reason why a riskless portfolio can be set up is because the stock price and the option price are both affected by the same underlying source of uncertainty. When an appropriate portfolio of the stock and the option is set up, the gain or loss from the stock position always offsets the gain or loss from the option position so that the overall value of the portfolio at the end of the short period of time is known with certainty.

There is one important difference between the Black-Scholes analysis and our analysis using a binomial model in Chapter 9. In Black-Scholes the position that is set up is riskless for only a very short period of time. To remain riskless it must be adjusted or rebalanced frequently. It is nevertheless true that the return from the riskless portfolio in any very short period of time must be the risk-free interest rate. This is the key element in the Black-Scholes arguments and leads to their pricing formulas.

Assumptions

The assumptions we use to derive the Black-Scholes differential equation are as follows:

1. The stock price follows the process developed in chapter 10 with μ and σ constant.
2. The short selling of securities with full use of proceeds is permitted.
3. There are no transaction costs or taxes. All securities are perfectly divisible.
4. There are no dividends during the life of the derivative.
5. There are no riskless arbitrage opportunities.
6. Security trading is continuous.
7. The risk-free rate of interest is constant and the same for all maturities.

11.5 Derivation of the Black-Scholes Differential Equation

We assume that the stock price follows the next process:

$$dS = \mu S dt + \sigma S dz$$

The variable f must be some function of S and t . Hence

$$df = \left(\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S} \sigma S dz$$

The discrete versions of these equations are

$$\Delta S = \mu S \Delta t + \sigma S \Delta z$$

$$\Delta f = \left(\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t + \frac{\partial f}{\partial S} \sigma S \Delta z$$

The appropriate portfolio is

$$\begin{aligned} & -1: \text{ derivative (short)} \\ & + \frac{\partial f}{\partial S}: \text{ shares (long)} \end{aligned}$$

Define Π as the value of the portfolio. By definition

$$\Pi = -f + \frac{\partial f}{\partial S} S \quad (11.12)$$

The change $\Delta \Pi$ in the value of the portfolio in time Δt is given by

$$\Delta \Pi = -\Delta f + \frac{\partial f}{\partial S} \Delta S$$

Substituting yields

$$\Delta \Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t \quad (11.14)$$

Since this equation does not involve Δz , the portfolio Π must be riskless during time Δt . It follows that

$$\Delta \Pi = r \Pi \Delta t$$

Substituting from equations (11.12) and (11.14), this becomes

$$\left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t = r \left(f - \frac{\partial f}{\partial S} S \right) \Delta t$$

So that

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \quad (11.15)$$

Equation (11.15) is the Black-Scholes differential equation. The particular derivative that is obtained when the equation is solved depends on the *boundary conditions* that are used. These specify the values of the derivative at the boundaries of possible values of S and t . In the case of a European call option, the key boundary condition is

$$f = \max(S - X, 0) \quad \text{when } t = T$$

In the case of a European put option, it is

$$f = \max(X - S, 0) \quad \text{when } t = T$$

11.6 Risk-Neutral Valuation

Risk-neutral valuation arises from one key property of the Black-Scholes differential equation (11.15). This property is that the equation does not involve any variables that are affected by the risk preferences of investors. The fact that the Black-Scholes differential equation is independent of risk preferences enables an ingenious argument to be used. If risk preferences do not enter the equation, they cannot affect its solution. Any set of risk preferences can therefore be used when evaluating f .

In a world where investors are risk neutral, the expected return on all securities is the risk-free rate of interest. It is also true that the present value of any cash flow in a risk-neutral world can be obtained by discounting its expected value at the risk-free rate.

It is important to realize that the risk-neutrality assumption is merely an artificial device for obtaining solutions to the Black-Scholes differential equation. The solutions that are obtained are valid in all worlds. When we move from a risk-neutral world to a risk-averse world, two things happen. The expected growth rate in the stock price changes and the discount rate that must be used for any payoffs from the derivative changes. It happens that these two effects offset each other exactly.

11.7 Black-Scholes Pricing Formulas

The expected value of a European call option at maturity in a risk-neutral world is

$$E[\max(S_T - X, 0)]$$

Where E denotes expected value in a risk-neutral world. From the risk-neutral valuation argument the European call option price, c , is the value of this discounted at the risk-free rate of interest, that is,

$$c = e^{-r(T-t)} E[\max(S_T - X, 0)] \quad (11.20)$$

In a risk-neutral world, $\ln S_T$ has the probability distribution in equation (11.2) with μ replaced by r ; that is,

$$\ln S_T \sim \phi \left[\ln S + \left(r - \frac{\sigma^2}{2} \right) (T-t), \sigma \sqrt{T-t} \right]$$

Evaluating the right-hand side of equation (11.20) is an application of integral calculus. The result is

$$c = SN(d_1) - Xe^{-r(T-t)} N(d_2) \quad (11.22)$$

Where

$$d_1 = \frac{\ln(S/X) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln(S/X) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

And $N(x)$ is the cumulative probability distribution function for a variable that is normally distributed with a mean of zero and a standard deviation of 1. Equation (11.22) can be written

$$c = e^{-r(T-t)} [SN(d_1)e^{r(T-t)} - XN(d_2)]$$

The expression $N(d_2)$ is the probability that the option will be exercised in a risk-neutral world so that $XN(d_2)$ is the strike price times the probability that the strike price will be paid. The expression $SN(d_1)e^{r(T-t)}$ is the expected value of a variable that equals S_T if $S_T > X$ and zero otherwise in a risk-neutral world.

Since $c = C$, equation (11.22) also gives the value of an American call option on a non-dividend-paying stock. The value of a European put can be calculated in a manner similar to a European call. Alternatively, put-call parity can be used. The result is

$$p = Xe^{-r(T-t)}N(-d_2) - SN(-d_1) \quad (11.23)$$

Unfortunately, no exact analytic formula for the value of an American put option on a non-dividend-paying stock has been produced. Note that to derive equations (11.22) and (11.23), it has been assumed that r is constant.

11.8 Cumulative Normal Distribution Function

A polynomial approximation can be used. One such approximation that can easily be obtained using a hand calculator is

$$N(x) = \begin{cases} 1 - N'(x)(a_1k + a_2k^2 + a_3k^3) & \text{when } x \geq 0 \\ 1 - N(-x) & \text{when } x < 0 \end{cases}$$

Where

$$k = \frac{1}{1 + \gamma x}$$

$$\gamma = 0.33267$$

$$a_1 = 0.4361836$$

$$a_2 = -0.1201676$$

$$a_3 = 0.9372980$$

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

This provides values for $N(x)$ that are usually accurate to four decimal places and are always accurate to 0.00002.

For six-decimal-place accuracy, the following can be used:

$$N(x) = \begin{cases} 1 - N'(x)(a_1k + a_2k^2 + a_3k^3 + a_4k^4 + a_5k^5) & \text{when } x \geq 0 \\ 1 - N(-x) & \text{when } x < 0 \end{cases}$$

Where

$$k = \frac{1}{1 + \gamma x}$$

$$\gamma = 0.2316419$$

$$a_1 = 0.319381530$$

$$a_2 = -0.356563782$$

$$a_3 = 1.781477937$$

$$a_4 = -1.821255978$$

$$a_5 = 1.330274429$$

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

11.10 Implied Volatilities

The one parameter in the Black-Scholes pricing formulas that cannot be observed directly is the volatility of the stock price. At this stage it is appropriate to mention an alternative approach that uses what is termed as *implied volatility*. This is the volatility implied by an option price observed in the market.

The implied volatility is the value of σ , which when substituted into equation (11.22) gives the value of c . Unfortunately, it is not possible to invert equation (11.22) so that σ is expressed as a function of $S, X, r, T-t$, and c . However, an iterative search procedure can be used to find the implied σ .

Implied volatilities can be used to monitor the market's opinion about the volatility of a particular stock. This does change over time. They can also be used to estimate the price of one option from the price of another option. Very often, several implied volatilities are obtained simultaneously from different options on the same stock and a composite implied volatility for the stock is then calculated by taking a suitable weighted average of the individual implied volatilities. The amount of weight given to each implied volatility in this calculation should reflect the sensitivity of the option price to the volatility.

11.11 The Causes of Volatility

Some analysts have claimed that the volatility of a stock price is caused solely by the random arrival of new information about the future returns from the stock. Others have claimed that volatility is caused largely by trading. An interesting question, therefore, is whether the volatility of an exchange-traded instrument is the same when the exchange is open as when it is closed. The results suggest that volatility is far larger when the exchange is open than when it is closed. Proponents of the view that volatility is caused only by new information might be tempted to argue that most new information on stocks arrives during trading days. The only reasonable conclusion seems to be that volatility is to some extent caused by trading itself.

What are the implications of all of this for the measurement of volatility and the Black-Scholes model? If daily data are used to measure volatility, the results suggest that days when the exchange is closed should be ignored. The volatility per annum can then be calculated from the volatility per trading day using the formula

$$\text{Volatility per annum} = \text{volatility per trading day} \times \sqrt{\text{number of trading days per annum}}$$

The normal assumption in equity markets is that there are 252 trading days per year.

Although volatility appears to be a phenomenon that is related largely to trading days, interest is paid by the calendar day. This has led to suggestions about option valuation, having two time measures to make the calculations:

$$\tau_1 : \frac{\text{trading days until maturity}}{\text{trading days per year}}$$

$$\tau_2 : \frac{\text{calendar days until maturity}}{\text{calendar days per year}}$$

And that the Black-Scholes formulas should be adjusted to

$$c = SN(d_1) - Xe^{-r\tau_2} N(d_2)$$

$$p = Xe^{-r\tau_2} N(-d_2) - SN(-d_1)$$

Where

$$d_1 = \frac{\ln(S/X) + r\tau_2 + \sigma^2\tau_1/2}{\sigma\sqrt{\tau_1}}$$

$$d_2 = \frac{\ln(S/X) + r\tau_2 - \sigma^2\tau_1/2}{\sigma\sqrt{\tau_1}} = d_1 - \sigma\sqrt{\tau_1}$$

In practice, this adjustment makes little difference except for very short life options.

11.12 Dividends

We assume that the amount and timing of the dividends during the life of an option can be predicted with certainty. A dividend-paying stock can reasonably be expected to follow the stochastic process developed in Chapter 10 except when the stock goes ex-dividend. At this point the stock's price goes down by an amount reflecting the dividend paid per share. For tax reasons, the stock price may go down by somewhat less than the cash amount of the dividend. To take account of this, the word *dividend* in this section should be interpreted as the reduction in the stock price on the ex-dividend date caused by the dividend.

European Options

European options can be analyzed by assuming that the stock price is the sum of two components: a riskless component that corresponds to the known dividends during the life of the option and a risky component. The riskless component at any given time is the present value of all the dividends during the life of the option discounted from the ex-dividend dates to the present at the risk-free rate. By the time the option matures, the dividends will have been paid and the riskless component will no longer exist. The Black-Scholes formula is therefore correct if S is put equal to the risky component of the stock price and σ is the volatility of the process followed by the risky component (in theory this is not quite the same as the volatility of the stochastic process followed by the whole stock price. The volatility of the risky component is approximately equal to the volatility of the whole stock price multiplied by $S/(S-V)$, where V is the present value of the dividends. In practice, the two are often assumed to be the same). Operationally, this means that the Black-Scholes formula can be used provided that the stock price is reduced by the present value of all dividends during the life of the option, the discounting being done from the ex-dividend dates at the risk-free rate.

American Options

When there are dividends, it is optimal to exercise only at a time immediately before the stock goes ex-dividend. We assume that n ex-dividend dates are anticipated and that t_1, t_2, \dots, t_n are moments in time immediately prior to the stock going ex-dividend with $t_1 < t_2 < \dots < t_n$. The dividends corresponding to these times will be denoted by D_1, D_2, \dots, D_n , respectively.

If the option is exercised at time t_n , the investor receives $S(t_n) - X$. If the option is not exercised, the stock price drops to $S(t_n) - D_n$. The value of the option is then greater than

$$S(t_n) - D_n - Xe^{-r(T-t_n)}$$

It follows that if

$$S(t_n) - D_n - Xe^{-r(T-t_n)} \geq S(t_n) - X$$

That is

$$D_n - Xe^{-r(T-t_n)} \leq X(1 - e^{-r(T-t_n)})$$

It cannot be optimal to exercise at time t_n . On the other hand, if

$$D_n > X(1 - e^{-r(T-t_n)})$$

For any reasonable assumption about the stochastic process followed by the stock price, it can be shown that it is always optimal to exercise at time t_n for a sufficiently high value of $S(t_n)$.

Consider next time t_{n-1} . If the option is exercised at this time, the investor receives $S(t_{n-1}) - X$. If the option is not exercised, a lower bound to the option price at that time is $S(t_{n-1}) - D_{n-1} - Xe^{-r(t_n - t_{n-1})}$. It follows that if

$$S(t_{n-1}) - D_{n-1} - Xe^{-r(t_n - t_{n-1})} \geq S(t_{n-1}) - X$$

or

$$D_{n-1} \leq X(1 - e^{-r(t_n - t_{n-1})})$$

It is not optimal to exercise at time t_{n-1} . Similarly, for any $i < n$, if

$$D_i \leq X(1 - e^{-r(t_{i+1} - t_i)}) \quad (11.26)$$

It is not optimal to exercise at time t_i .

The inequality in (11.26) is approximately equivalent to

$$D_i \leq Xr(t_{i+1} - t_i)$$

We can conclude from this analysis that in most circumstances, the only time that needs to be considered for the early exercise of an American call is the final ex-dividend date, t_n .

Black's Approximation

Black suggests an approximate procedure for taking account of early exercise. This involves calculating, as described earlier in this section, the prices of European options that matures at time T and t_n , and the setting the American price equal to the greater of the two.

Up to now, our discussion has centered around American call options. The results for American put options are less clear cut. Dividends make it less likely that an American put option will be exercised early. It can be shown that it is never worth exercising an American put for a period immediately prior to an ex-dividend date. Indeed, if

$$D_i \leq X(1 - e^{-r(t_{i+1} - t_i)})$$

For all $i < n$ and

$$D_n \geq X(1 - e^{-r(T-t_n)})$$

An argument analogous to that just given shows that the put option should never be exercised early.