CHAPTER 7: *PROPERTIES OF STOCK OPTION PRICES*

7.1 Factors Affecting Option Prices

Table 7.1 Summary of the Effect on the Price of a Stock Option of Increasing One Variable While Keeping All Other Fixed

Variable	European Call	European Put	American Call	American Put
Stock price	+	-	+	-
Strike price	-	+	-	+
Time to expiration	?	?	+	+
Volatility	+	+	+	+
Risk-free rate	+	-	+	-
Dividends	-	+	-	+

Time to expiration

European put and call options do not necessarily become more valuable as the time to expiration increases. This is because it is not true that the owner of a long-life European option has all the exercise opportunities open to the owner of a short-life European option.

Volatility

As volatility increases, the chance that the stick will do very well or very poorly increases. For the owner of a stock, these two outcomes tend to offset each other. However, this is not so for the owner of a call or put.

Risk-Free Interest Rate

As interest rate in the economy increase, the expected growth rate of the stock price tends to increase. However, the present value of any future cash flows received by the holder of the option decreases. These two effects both tend to decrease the value of a put option. Hence, put option prices decline as the risk-free interest rate increases. In the case of calls, the first effect tends to increase the price while the second effect tends to decrease it. It can be shown that the first effect always dominates the second effect; that is, the prices of calls always increase as the risk-free interest rate increases.

It should be emphasized that these results assume that all other variables remain fixed. In practice when interest rates rise (fall), stock prices tend to fall (rise). The net effect of an interest rate change and the accompanying stock price change may be different from that just given.

7.2 Assumptions and Notation

We assume that there are some market participants for which

1. There are no transactions costs.
2. All trading profits (net of trading losses) are subject to the same tax rate.
3. Borrowing and lending at the risk-free interest rate is possible.

We assume that these market participants are prepared to take advantage of arbitrage opportunities as they arise. For the purpose of our analyses, it is therefore reasonable to assume that there are no arbitrage opportunities. We will use the following notation:

S : current stock price

X : strike price of option

T : time of expiration of option

t : current time

S_T : stock price at time T

r : risk-free rate of interest for maturity T (continously compounded)

C : value of American call option to buy one share

P : value of American put option to sell one share

c : value of European call option to buy one share

p : value of European put option to sell one share

σ : volatility of stock price

7.3 Upper and Lower Bounds for Option Prices

Upper Bound

An American or European call option gives the holder the right to buy one share of a stock for a certain price. No matter what happens, the option can never be worth more than the stock. Hence,

$$c \le S \text{ and } C \le S$$

An American or European put option gives the holder the right to sell one share of a stock for X. No matter how low the stock price becomes, the option can never be worth more than X. Hence

$$p \le X \text{ and } P \le X$$

For European options, we know that at time T, the option will not be worth more than X. It follows that it must now not be worth more than the present value of X:

$$p \le Xe^{-r(T-t)}$$

Lower Bound for Calls on Non-Dividend-Paying Stocks

A lower bound for the price of a European call option on a non-dividend-paying stock is

$$S - Xe^{-r(T-t)}$$

For a formal argument, we consider the following two portfolios:

Portfolio A: one European call option plus an amount of cash equal to $Xe^{-r(T-t)}$.
Portfolio B: one share.

In portfolio A, if the cash is invested at the risk-free interest rate, it will grow to X at time T. If $S_T > X$, the call option is exercised at time T and portfolio A is worth S_T. If $S_T < X$, the call option expires worthless and the portfolio is worth X. Hence, at time T, portfolio A is worth $\max(S_T, X)$. Portfolio B is worth S_T at time T. Hence, portfolio A is always worth as much as, and is sometimes worth more than, portfolio B at time T. It follows that it must be worth more than portfolio B today. Hence

$$c + Xe^{-r(T-t)} > S \quad \text{or} \quad c > S - Xe^{-r(T-t)}$$

Since the worst that can happen to a call option is that it expires worthless, its value must be positive. This means that c>0 and therefore

$$c > \max\left(S - Xe^{-r(T-t)}, 0\right)$$

Lower Bound for European Puts on Non-Dividend-Paying Stocks

For a European put option on a non-dividend-paying stock, a lower bound for the price is

$$Xe^{-r(T-t)} - S$$

Portfolio C: one European put option plus one share
Portfolio D: an amount of cash equal to $Xe^{-r(T-t)}$

If $S_T < X$, the option in portfolio C is exercised at time T and the portfolio becomes worth X. If $S_T > X$, the put option expires worthless and the portfolio is worth S_T at time T. Hence portfolio C is worth $\max(S_T, X)$ at time T. Assuming that the cash is invested at the risk-free interest rate, portfolio D is worth X at time T. Hence, portfolio C is always worth as much as, and is sometimes worth more than, portfolio D at time T. It follows that in the absence of arbitrage opportunities, portfolio C must be worth more than portfolio D today. Hence

$$p + S > Xe^{-r(T-t)} \quad \text{or} \quad p > Xe^{-r(T-t)} - S$$

Since the worth that can happen to a put option is that it expires worthless, its value must be positive. This means that

$$p > \max\left[Xe^{-r(T-t)} - S, 0\right]$$

7.4 Early Exercise: Calls on a Non-Dividend-Paying Stock

In this section we show that it is never optimal to exercise an American call option on a non-dividend-paying stock early. To present a formal argument, consider again the following two portfolios:

Portfolio E: one American call option plus an amount of cash equal to $Xe^{-r(T-t)}$.
Portfolio F: one share

The value of the cash in portfolio E at expiration of the option is X. At some earlier time τ, it is $Xe^{-r(T-\tau)}$. If the call option is exercised at time τ, the value of portfolio E is $S - X + Xe^{-r(T-\tau)}$. This is always less than S when $\tau < T$ since r>0. Portfolio E is therefore always worth less than portfolio F if the call option is exercised prior to maturity. If the call option is held to expiration, the value of portfolio E at time T is $\max(S_T, X)$. The value of portfolio F is S_T. There is always some change that $S_T < X$. This means that portfolio E is always worth as much as, and is sometimes worth more than, portfolio F.

We have shown that portfolio E is worth less than portfolio F if the option is exercised immediately, but is worth at least as much as portfolio F if the holder of the option delays exercise until the expiration date. An American call option on a non-dividend-paying stock is therefore worth the same as the corresponding European call option on the same stock:

$$C = c$$

$$c > S - Xe^{-r(T-t)}$$

$$C > S - Xe^{-r(T-t)}$$

Figure 7.1 shows the general way in which the call price varies with S. As r, σ or $T-t$ increase, the call price moves in the direction indicated by the arrows.

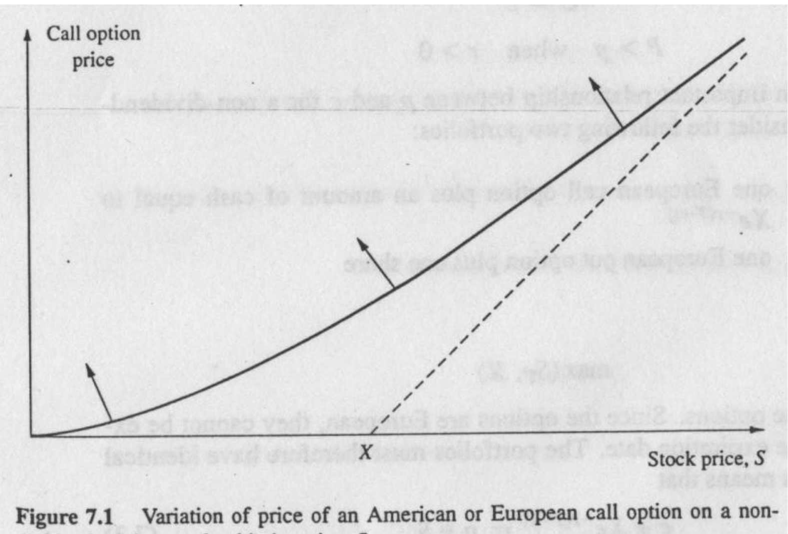

Figure 7.1 Variation of price of an American or European call option on a non-dividend-paying stock with the price, S.

To summarize, one reason why a call option should not be exercised early can be considered as being due to the insurance that it provides. Another reason is concerned with the time value of money. The latter the strike price is paid out the better.

7.5 Early Exercise: Puts on a Non-Dividend-Paying Stock

It can be optimal to exercise an American put option on a non-dividend-paying stock early.

Portfolio G: one American put option plus one share
Portfolio H: an amount of cash equal to $Xe^{-r(T-t)}$

If the option is exercised at time $\tau < T$, portfolio G becomes worth X while portfolio H is worth $Xe^{-r(T-\tau)}$. Portfolio G is therefore worth more than portfolio H. If the option is held to expiration, portfolio G becomes worth $\max(S_T, X)$ while portfolio H is worth X. Portfolio G is therefore worth at least as much as, and possible more than, portfolio H. Here we cannot argue that early exercise is undesirable since portfolio G looks more attractive than portfolio H regardless of the decision on early exercise.

Like a call option, a put option can be viewed as providing insurance. However, a put option is different from a call option in that it may be optimal for an investor to forgo this insurance and exercise early in order to realize the strike price immediately. In general, the early exercise of a put option becomes more attractive as S decreases, as r increases, and as σ decreases.

It will be recalled from equation $p > \max\left[Xe^{-r(T-t)} - S, 0\right]$ that $p > Xe^{-r(T-t)} - S$. For an American put with price P, the stronger condition $P \geq X - S$ must always hold since immediate exercise is always possible.

Figure 7.2 shows the general way in which the price of an American put varies with S. The curve representing the value of the put therefore merges into the put's intrinsic value, X-S, for a sufficiently small value of S. The value of the put moves in the direction indicated by the arrows when r decreases, when σ increases, and when T increases.

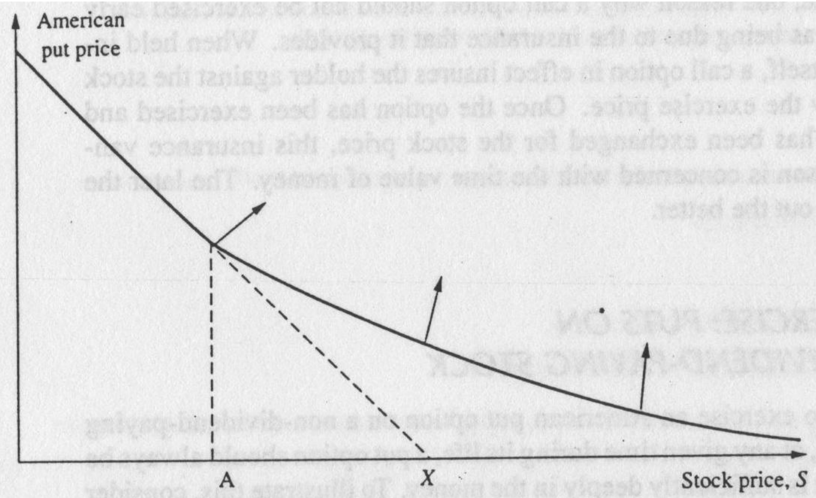

Figure 7.2 Variation of price of an American put option with the stock price, S.

Since there are some circumstances when it is desirable to exercise an American put option early, it follows that an American put option is always worth more than the corresponding European put option. Since an American put is sometimes worth its intrinsic value, it follows that a European put option must sometimes be worth less than its intrinsic value. Figure 7.3 shows the variation of the European put price with the stock price. Point E in figure 7.3 is where S=0 and the European put price is $Xe^{-r(T-t)}$.

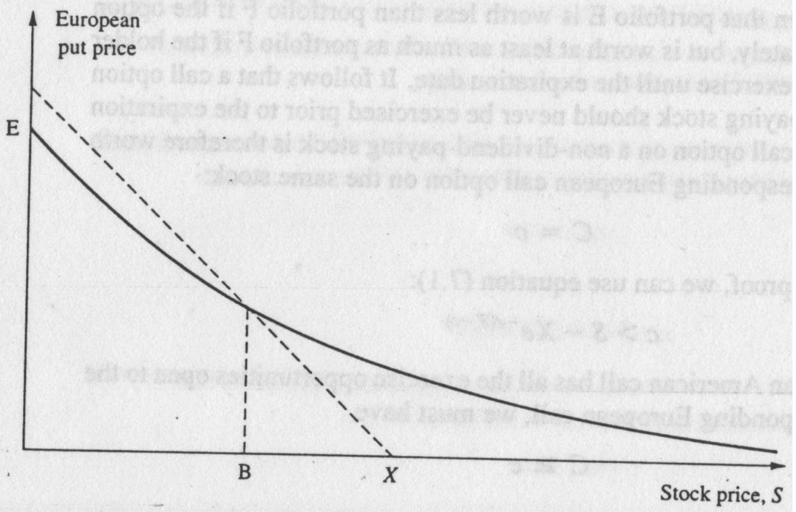

Figure 7.3 Variation of price of a European put option with the stock price, S.

7.6 Put-Call Parity

We have shown that for a non-dividend-paying stock, $C = c$ and $P > p$ when $r > 0$. We now derive an important relationship between *p* and *c* for a non-dividend-paying stock. Consider the following two portfolios:

Portfolio A: one European call option plus an amount of cash equal to $Xe^{-r(T-t)}$
Portfolio C: one European put option plus one share

Both are worth $\max(S_T, X)$ at the expiration of the options. The portfolio must therefore have identical values today: $c + Xe^{-r(T-t)} = p + S$. This relationship is known as *put-call parity*. It shows that the value of a European call with a certain exercise price and exercise date can be deduced from the value of a European put with the same exercise price and date, and vice versa.

Relationship between American Call and Put Prices

Put-call parity holds only for European options. However, it is possible to derive some relationships between American option prices for a non-dividend-paying stock. Since $P > p$, it follows from the call-put parity that $P > c + Xe^{-r(T-t)} - S$, and since c=C, $P > C + Xe^{-r(T-t)} - S$ or $C - P < S - Xe^{-r(T-t)}$. For a further relationship between C and P, consider

Portfolio I: one European call option plus an amount of cash equal to X
Portfolio J: one American put option plus one share

Both options have the same exercise price and expiration date. Assume that the cash in portfolio I is invested at the risk-free interest rate. If the put option is not exercised early portfolio J is worth $\max(S_T, X)$ at time T. Portfolio I is worth $\max(S_T, X) + Xe^{r(T-t)} - X$ at this time. Portfolio I is therefore worth more than portfolio J. Suppose next that the put option in portfolio J is exercised early, say, at time τ. This means that portfolio J is worth X at time τ. However, even if the call option were worthless, portfolio I would be worth $Xe^{r(T-\tau)}$ at time τ. It follows that portfolio I is worth more than portfolio J in all circumstances. Hence

$$c + X > P + S, \text{ since } c = C$$

$$C + X > P + S \text{ or } C - P > S - X$$

Combining this with $C - P < S - Xe^{-r(T-t)}$, we obtain

$$S - X < C - P < S - Xe^{-r(T-t)}$$

7.3 Effect of Dividends

In this section we discuss the impact of dividends. We will use D to denote the present value of the dividends during the life of the option. For this purpose a dividend is assumed to occur at the time of its ex-dividend date.

Lower Bound for Calls and Puts

We can redefine portfolios A and B as follows:

Portfolio A: one European call option plus an amount of cash equal to $D + Xe^{-r(T-t)}$
Portfolio B: one share

A similar argument to the one used to derive equation $c > \max\left(S - Xe^{-r(T-t)}, 0\right)$ shows that $c > S - D - Xe^{-r(T-t)}$. We can also redefine portfolios C and D as follows:

Portfolio C: one European put option plus one share.
Portfolio D: an amount of cash equal to $D + Xe^{-r(T-t)}$.

A similar argument to the one used to derive equation $p > \max\left[Xe^{-r(T-t)} - S, 0\right]$ shows that $p > D + Xe^{-r(T-t)} - S$.

Early Exercise

When dividends are expected, we can no longer assert that an American call option will not be exercised early. Sometimes it is optimal to exercise an American call immediately prior to an ex-dividend date. This is because the dividend will cause the stock price to jump down, making the option less attractive. It is never optimal to exercise a call at other times.

Put-Call Parity

Comparing the value at time T of the redefined portfolios A and C shows that when there are dividends, put-call parity becomes

$$c + D + Xe^{-r(T-t)} = p + S$$

Dividends cause equation $S - X < C - P < S - Xe^{-r(T-t)}$ to be modified to

$$S - D - X < C - P < S - Xe^{-r(T-t)}$$

To prove this inequity, consider

Portfolio I: one European call option plus an amount of cash equal to $D+X$
Portfolio J: one American put option plus a share

Regardless of what happens, it can be shown that portfolio I is worth more than portfolio J. Hence $P + S < c + D + X$. Since a European call is never worth more than its American counterpart, it follows that $P + S < C + D + X$ or $S - D - X < C - P$. This proves the first half of the inequity. For a non-dividend-paying stock, we showed in equation $S - X < C - P < S - Xe^{-r(T-t)}$ that $C - P < S - Xe^{-r(T-t)}$. Since dividends decrease the value of a call and increase the value of a put, this inequity must also be true for options on divided-paying stock. This proves the second half of the inequity.